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SELF-PROVING EXAMPLES IN ARITHMETIC.

ELLIS.

SELF-PROVING EXAMPLES

IN THE FOUR FIRST RULES OF

ARITHMETIC,

SIMPLE & COMPOUND,

ESPECIALLY ADAPTED

FOR SELF-PRACTICE AS WELL AS SCHOOL OR FAMILY USE:

ALLOWING THE TEACHER TO SET INNUMERABLE EXAMPLES SIMULTANEOUSLY, AND TO VERIFY THE RESULTS AT SIGHT, WITHOUT PERMITTING THE PUPILS TO FORESEE AND THEREFORE "FORCE THE ANSWER."

TO WHICH ARE ADDED, EXAMPLES IN

CONTRACTED DIVISION, SQUARE AND CUBE ROOT,

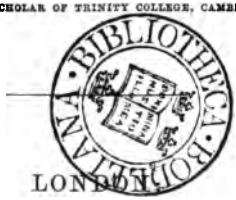
AS SELF-PRACTICE FOR ADVANCED COMPUTERS,

WITH AN EXPLANATION OF GUY'S AND HORNER'S RULES.

BY

ALEXANDER J. ELLIS, B.A.

FORMERLY SCHOLAR OF TRINITY COLLEGE, CAMBRIDGE.



LONGMAN, BROWN, GREEN, AND LONGMANS,
PATERNOSTER ROW.
1855.

181. c. 19.

PREFACE.

COMPUTATION, or the transformation of sums, differences, products, and quotients, into ordinary numbers, is a series of operations which can only be performed correctly and rapidly—and they are rarely, if ever, performed correctly if they are not performed rapidly—without long and severe practice. The number of examples given in ordinary works on arithmetic is so small as to be nearly valueless, and the ordinary artificial modes of forming examples require such a previous knowledge of arithmetic, either to set or to verify, as make them totally unfitted for *self-practice*, and comparatively inapplicable to school or family use. Even Mr. Tate's ingenious mode of setting examples—which has been introduced below in an extended and corrected form (*see App.* 3 and 4)—can, as he left it, only be used by the master, and the time which would be wasted by a teacher who has himself to write or dictate every question to his pupils, renders all such methods inapplicable to school use, and quite unfit for self-practice, however well-adapted for private teaching.

Impressed with these views, the author has for some years endeavoured to devise simple methods of setting examples which should be applicable both to school or family use

and self-practice, the latter being at least as important as the former. After trying a great variety of different contrivances, he is induced to offer the following to the public as at once simple and complete. They turn upon the properties of arithmetical complements, complete or diminished by 1, and numbers divisible by 9 9, all of which can be rapidly formed. Although most of the properties are so easily verified that they have only to be stated in algebraical symbols to be understood, the author has thought it best to give the theory in an appendix for the use of teachers acquainted with elementary algebra. To prevent any confusion, the portions intended for self-practice and school or family use, have been kept entirely distinct. The examples for the use of teachers in schools are equally well adapted for teachers in private families. Governesses and parents have already successfully used some of the methods suggested, which are very simple in practice, and effect a great saving of time and labour to the teacher. Teachers and self-practisers must not suppose that because they may have some little difficulty in comprehending the rules laid down, there is any difficulty in their application. The verbal description of some of our simplest movements is often exceedingly complicated, and although the author has aimed at brevity and perspicuity, taking care in all cases to give a fully explained example, he cannot flatter himself with the hope that he has been always successful in so difficult an operation. The book is published with a hope that it will be found equally useful for the laborious adult or youth who wishes to qualify himself for business as a ready and certain computer, and for the teacher in schools and private families

who wishes to exercise pupils thoroughly in the acquisition of this indispensable art.

The *Preliminary Exercises* are an attempt to furnish examples for self-practice in Professor De Morgan's admirable rules given in the *Companion to the Almanack* for 1844. The person who has patience to work them all, and never leave one till he can perform it with perfect accuracy and the utmost rapidity, will in six months make himself a perfect computer. Of course it will not follow that he can work examples occurring in the business of life unless he knows the science of arithmetic as well as the art of computation. The writer therefore recommends a particular study of some such work as Professor De Morgan's *Arithmetic* to give him the requisite knowledge.

For the benefit of advanced computers, exercises in contracted division, and the extraction of square and cube roots, have been annexed. M. Guy's admirable rule for contracted division being little, if at all, known in this country, is given at length for its commonest case; and Horner's rule for the extraction of cube root being still but sparingly employed, notwithstanding its excellence and Professor De Morgan's efforts to make it known, is also fully explained. Although such explanations are, strictly speaking, beyond the scope of this little treatise, their general absence from books on arithmetic will excuse their insertion as an introduction to the examples. Cube roots, it may be urged, have very rarely to be extracted in actual practice. This is perfectly true. But it is also true that the ability to extract them readily can only be possessed by an accomplished

computer, and hence no one who wishes to become such will neglect to exercise himself diligently in working examples of the kind here proposed according to the methods recommended, not for the sake of the actual operations, but for the mass of subsidiary processes which they call into play. On the other hand, it has not been deemed necessary to add examples in Vulgar Fractions and Decimals, as they involve no new processes requiring especial exercise.

"Calculate like a machine, understand like its maker," should be the computer's motto.

A. J. E.

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SELF-PROVING EXAMPLES

IN

ARITHMETIC.

DEFINITIONS, &c.

*** These definitions being all referred to as the words occur hereafter, Teachers or Self-Practisers may proceed at once to the *Preliminary Exercises*, p. 6, or *Auxiliary Table*, p. 14.

1. *Subcomplement.* Take each digit in a number from 9. The result is here called the *subcomplement* of that number. (*App.* **1.**) The self-instructed may form subcomplements by means of the Auxiliary Table (*see* **38**).

Ex. 562 is the subcomplement of 437, and, conversely, 437 is the subcomplement of 562; for if 5, 6 and 2 be each taken from 9, the results are 4, 3, and 7.

2. *Bīnōnal.* Any number followed by its subcomplement with or without the interposition of *nines*, is here called a *bīnōnal* or "double nine" number, because it divides into two parts, which, when added, produce a series of nines.

In *binōnal*, pronounce the first syllable as *by*, and the second to rhyme with *moan*.

Ex. 562437, 5629437, 56299437, &c., are binōnals, because 562 and 437, and also 0562 and 9437, and also 00562 and 99437 are subcomplementary (**1**).

3. *Base.* The first part of a binonal is its base.

Ex. 5 6 2, or 0 5 6 2, or 0 0 5 6 2 form the first part or base of the binōnals in the last example.

4. *Extent.* The *number* of digits in the *second* part of a binonal, including the interposed nines, is its extent.

Ex. The extent of 5 6 2 4 3 7 is 3, because its second part, 4 3 7, contains 3 digits. The extent of 5 6 2 9 4 3 7 is 4, because its second part, 9 4 3 7, contains 4 digits.

5. To form a binonal of any required extent not less than the number of digits in the base, we have therefore only to interpose the requisite number of *nines*. (*App.* **2** to **5**.)

Ex. Form binonals of the extent 6, from the bases 5 3, 2 6 7 4, 7, 3 8 5 4 3, 2 0 8 6 7 2. Results, 5 3 9 9 9 9 4 6, 2 6 7 4 9 9 7 3 2 5, 7 9 9 9 9 9 2, 3 8 5 4 3 9 6 1 4 5 6, 2 0 8 6 7 2 7 9 1 3 2 7.

6. *Complement.* The subcomplement of a number increased by 1 is its complement, and is therefore formed by taking each digit except the last on the right, from 9, and this last from 1 0. (*App.* **6**.)

Ex. Number 5 6 2, *sub*complement 4 3 7, *complement* 4 3 8.

7. *Supercomplement.* Increase each digit in the complement by 1, and the result is the supercomplement. (*App.* **7**.)

Ex. Number 5 6 2, complement 4 3 8, supercomplement 5 4 9.

8. To form the supercomplement, when no digit is less than 1, and the last on the right is not less than 2, subtract all but this last from 1 0, and this last from 1 1. When this condition is not fulfilled, subtract the number from a series of *ones*.

Ex. 1. Number 5 6 2 ; take 5 and 6 from 1 0, and 2 from 1 1, the results are 5, 4 and 9. Hence 5 4 9 is the supercomplement, as in (**7**).

Ex. 2. Number 2 0 1, here take 2 and 0 from 1 0, and 1 from 1 1, the results are 8, 1 0, and 1 0, which are not digits. Hence take 2 0 1 from 1 1 1 1, and the result is 9 1 0, which is its supercomplement, or its complement 7 9 9 increased by 1 1 1.

9. *Nōnal.* Any number divisible without remainder by 9 is here termed nōnal.

10. To form a nōnal, write down a binōnal. **(2.)**

11. *Last sum.* Add the digits of any number. Express the result as a number, and add *its* digits. And so on till a number is reached containing only one digit. This is the last sum. It is the remainder after dividing the number by 9. Hence, if the last sum is 9, the number is nonal. (*App.* **8.**)

Ex. Number 5 6 2 ; the sum of the digits 5, 6 and 2 is 1 3 ; the sum of the digits 1 and 3 of this sum is 4. Hence 4 is the last sum, or remainder after dividing 562 by 9 (the quotient in this case would have been 6 2, but is not wanted), and consequently 5 6 2 is not nonal. But 5 6 7 gives for the first sum of its digits 1 8, and the last sum 9, hence 5 6 7 is nonal.

12. *Undecimal.* A number divisible without remainder by 11 is here called undecimal.

13. To form an undecimal number, form a nonundecimal. **(15.)**

14. *Last difference.* Subtract the first left hand digit of any number from the second, and the difference from the second digit, and this difference from the next digit. When any subtraction would be impossible, borrow 10 and carry 1 to the difference before performing the next subtraction, or if it is the last difference simply increase it by 1. The last difference thus obtained is called the last difference of the number. It is the remainder after dividing the number by 11, and if it is 0, the number is undecimal. (*App.* **9.**)

Ex. 1. The last difference of 8 5 6 3 2 8 4 is 4, which is found thus. Take 8, the first digit, from 1 5 (the second digit 5 increased by 1 0), the difference is 7, to which we must carry 1 (by the rule) before the next subtraction; then take the result, 8, from 1 6 (the third digit increased by 1 0) leaving 8, which increased by 1, on account of the borrowed 1 0, is 9 ; then 9 from 1 3, 4 ; 5 from 1 2, 7 ; 8 from 8, 0,

and 0 from 4, 4, which is the last difference required. This process may, by a little practice, be performed as rapidly as a common subtraction, with which it is in fact identical. The computer should only say, 8, 7; 8, 8; 9, 4; 5, 7; 8, 0; 0, 4.

Ex. 2. Of 8 5 6 3 2 8 4 4, the same number as before, with its last difference added on as a last digit, the last difference will of course be 0, and this gives a second and generally simple way of forming an undecimal number.

Ex. 3. Of 8 5 6, the last difference is 9; found thus, 8 from 15, 7; 8 from 16, 8; and 8 increased by 1, because of the borrowed 10, is 9.

15. *Nōnundecimal.* A number divisible by both 9 and 11—that is, by 99—without remainder, is here called nonundecimal. (*App.* **10** to **15.**)

Pronounce the first syllable to rhyme with *moan*, and not like *non*.

16. To form a nonundecimal, write down a binonal with an *even* extent; that is, with an *even* number of digits in its second part. (*App.* **16.**)

Ex. 2376, 297, 2359764, 37846215, 28563971436, &c., are nonundecimal, because their extents are *even*, being 2, 4 or 6.

17. We recognise a number to be nonundecimal, first, when it does not exceed 4 digits, by seeing that it is binonal (*App.* **17**), and secondly, when it exceeds 4 digits, by seeing that its last sum (**11**) is 9, and its last difference (**14**) is 0.

Ex. 1. 297, 2376, 9801 are binonal, and hence, as they have no more than 4 digits, also nonundecimal, being in fact respectively 99 times 3, 24 and 99.

Ex. 2. 11880 is also nonundecimal, for its first sum is 18 and last sum 9, and its differences are 0, 8, 0, 0, the last being 0. But 11880 is *not* binonal.

18. *Proof.* When the result of an operation satisfies certain conditions, it may show that the whole work has been absolutely correct, or may give a strong presumption that it is so, sufficient to satisfy any student or teacher who is working or examining examples for practice. This satisfaction of conditions is termed a *proof*. (*App.* **18.**)

19. In all multiplications, if we take the last sums of the multiplicand and multiplier, and multiply them, the last sum of *this* product will be the same as the last sum of the complete product. The same is true for last differences.

In all divisions, if we take the last sums of the divisor and quotient, and multiply them, adding the result to the last sum of the remainder, and take the last sum of this result, it will be the same as the last sum of the dividend. The same is true for last differences.

This, then, is a *proof* for *all* multiplications and divisions, sufficient, when *both* last sums and last differences are used, to make the computer morally certain of the correctness of his result. (*App.* **19.**)

Ex. 1. Multiplication.

The product of 2576 by 328 is			844928
Last sums	2	4	8
Last differences	2	9	7

Product of last sums 8, the last sum of which, 8, is that of the complete product.

Product of last differences 18, the last difference of which, 7, is that of the original product.

Ex. 2.	Divisor.	Quotient.	Remainder.	Dividend.
	327	78	263	25769
Last sums	3	6	2	2
Last differences	8	1	10	7

Product of the last sums of divisor and quotient 18, and its last sum 9, added to 2, the last sum of remainder, gives 11, the last sum of which is 2, being that of dividend.

Product of the last differences of divisor and quotient 8, and its last difference 8, added to 10, the last difference of remainder, gives 18, the last difference of which is 7, being that of dividend.

Note. The usual proof by "casting out the nines" is the same as the proof by "last sums." It is, however, very imperfect, as the order of the digits in any number may be altered at pleasure without altering the last sum. The proof by "casting out the elevens," or by "last differences," is, on the other hand, so complete, since it will indicate any error which does not affect two adjacent digits in precisely the same degree, that it may be considered practically sufficient as a voucher for the correctness of products and quotients.

PRELIMINARY EXERCISES.

20. The following exercises are founded upon those recommended by Professor De Morgan, in his paper on *Arithmetical Computation*, in the *Companion to the Almanack* for 1844. The chief novelty is in the easy verifications appended to each exercise, which makes them convenient for self-practice without the aid of a master.

These preliminary exercises should be practised *daily*. In school and with a class, Tables 1 and 2 should be written out on a chart, or painted on a black board, and each boy in succession should perform one of the following exercises as rapidly as possible, the ear of the master being sufficient to verify the results. Where the Teacher, however, prefers to exercise mentally, without any use of the eye, he will find it useful to copy the columns of Table 2 on slips of pasteboard, making the figures rather large. He will then be able to use any column for calling out the figures, and by placing any two or more columns in juxta-position can work out the exercises hereafter described, as will be more particularly noticed in the proper place.

The self-practiser will write down all his results as rapidly as possible, and verify them as hereafter pointed out.

☞ No day should pass without devoting at least five minutes to these preliminary exercises,—the true sol-fa of computation, sufficient without further examples to make a rapid and certain computer. The most important exercises are those in (**23**) and (**24**) for Addition, (**27**) for Subtraction, and (**30**) and (**31**) for Multiplication. The two latter cannot be sufficiently practised, as they are the form in which the multiplication table is really applied, the added number being the digit carried. The object of the exercise in (**32**) is to enable the pupil to multiply by numbers between 10 and 20 in one line, adding in the back figure as well as the carried digit, or to add a product to another number, as in the condensed form for Cube Root, p. 56. It may therefore be omitted by those who do not intend to avail themselves of these condensations. Similarly the exercises for Long Division are intended to allow the computer to write down the remainders at once, as in (**63**), and may therefore be omitted by those who are satisfied with the ordinary process. The exercise for Short Division (**33**) only really differs from that in (**31**) by requiring the pupil to see a *difference* instead of a *sum*.

The Teacher will find the theory of the exercises in *App.* (**20.**)

21.—TABLE I.

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

22.—TABLE II.

1	2	3	4	5	6	7	8	9
3	8	5	8	7	1	5	2	4
5	6	9	1	9	3	3	1	8
7	4	6	5	1	5	1	3	5
9	5	5	9	3	8	9	5	1
2	1	7	2	4	2	5	6	3
4	3	1	6	6	9	8	7	6
6	7	4	3	5	5	2	5	2
8	5	2	5	2	4	4	9	7
5	9	8	7	8	7	6	4	5

Addition.

23. Take any digit and add to it successively (and *vivâ voce* without writing down the results) all the digits in any column of Table 1. The result is always a number of two digits, of which the first is the digit continually added, and the last is the digit with which the pupil commenced.

Ex. Take 7 and add to it all the 8's in Table 1, column 8. The results are 7, 15, 23, 31, 39, 47, 55, 63, 71, 79, 87. Here 8 (the first digit) is the number continually added, and 7 (the second digit) the number with which the operation commenced.

24. Add one and the same number to each of the digits in any column in Table 2, and add the results. The sum will be 10 times the result for the line containing *5*.

Ex. Add 7 to each digit in Table 2, col. 3; the results are 10, 12, 16, 13, 12, 14, 8, 11, 9, 15, and the sum of them is 120 or 10 times 12, the result from the line containing 5.

25. Add the digits in any two adjacent columns in Table 2 *sideways*. The sum of the results is 100.

Ex. Take the second and third columns, and add 2 and 3, then 8 and 5, then 6 and 9, and so on; the results are 5, 13, 15, 10, 10, 8, 4, 11, 7, 17, the sum of which is 100.

Mental Practice.

25^b. The Teacher taking *two* pasteboard slips of columns in Table 2, calls out all the numbers in succession, with the exception of any one of them (as, for instance, the last number on the *first* slip), and the pupils form the results mentally. On their naming the final result, the Teacher has only to add it to the omitted number in order to produce 100.

Ex. Suppose the slips to contain cols. 1 and 2, and the number omitted to be 5, the last digit of the first col., then the sum of the numbers given out will be 95.

It is better to omit a digit from the first slip, as otherwise the sum of the first set of numbers being always 50, the effect will be too artificial.

25^c. Take two slips and place them side by side. Read out the two digits in each line as tens and units, and get the pupils to add the results mentally. Omit *one line* beginning with *5*. Then the sum will be 500 less the units in the omitted line.

Ex. Taking the two first cols., the Teacher gives out, 12, 38, 56, 74, 95, 21, 43, 67, 85, and omits 59, the sum will be 491, or 500 less 9, the units in the omitted 59. If the pupils find it too difficult at first to add, for example, 38 to 12, begin by calling out

the tens and then the units, thus, 12, 30, 8, 50, 6, 70, 4, 90, 5, 20, 1, 40, 3, 60, 7, 80, 5. This will show them the method they are to pursue in the complete mental addition, *beginning* with the tens and then adding in the units.

Subtraction.

26. Select any number of two digits, as 87, and subtract the first digit from it as often as it is contained in the corresponding column of Table 1. The result is the second digit.

Ex. Subtracting 8, the first digit of 87, from 87 successively, we have 79, 71, 63, 55, 47, 39, 31, 23, 15, 7. The result (7) is the second digit of 87.

This should be done with great rapidity, and without writing down the partial results.

27. Subtract one and the same digit in succession from each digit in any column of Table 2, writing down the results, and borrowing 10 when necessary. When the constant subtrahend does not exceed 5, the sum of the remainders is 40, but otherwise it is 50.

Ex. From col. 3,			or...	3,	5,	9,	6,	5,	7,	1,	4,	2,	8	
Take 1;	remainders...	2,	4,	8,	5,	4,	6,	0,	3,	1,	7,	sum	40
,, 2	,,	...	1,	3,	7,	4,	3,	5,	9,	2,	0,	6	,,	40
,, 5	,,	...	8,	0,	4,	1,	0,	2,	6,	9,	7,	3	,,	40
,, 7	,,	...	6,	8,	2,	9,	8,	0,	4,	7,	5,	1	,,	50
,, 9	,,	...	4,	6,	0,	7,	6,	8,	2,	5,	3,	9	,,	50

This example is well adapted for mental practice. The constant subtrahend being named, the Teacher takes up a slip containing one of the cols. of Table 2, and calls out the numbers in order, the pupil names the difference, borrowing 10 where necessary. As each result is named at once, the teacher relies upon himself for checking each result as it arises. But as much time is lost by speaking, it is better to train the eye only. The Teacher *names* the constant subtrahend, and merely *points* to the minuend, which the pupil answers instantly. In this case, as the Teacher uses the Table, he is recommended not to point to consecutive figures, in order that the pupil may have no time for preparation.

28. From each digit in any column in Table 2, subtract the corresponding digit in any other column, write down the remainders, and the borrowed tens in separate lines. The sum of the remainders will be the sum of the borrowed tens.

Ex. Take columns 2 and 3.

From	3,	5,	9,	6,	5,	7,	1,	4,	2,	8	
Take	2,	8,	6,	4,	5,	1,	3,	7,	5,	9	
Remainders...	1,	7,	3,	2,	0,	6,	8,	7,	7,	9	sum 50
Borrowed.....		10,					10,	10,	10,	10	sum 50

Only the last line but one is written in actual practice, and the fact of a ten being borrowed is intimated by a dash.

If the Teacher employs the pasteboard slips, he may conveniently use this process for mental practice, but the Teacher is recommended to cultivate the *eye* rather than the ear in subtraction. It is best not to name any number but the *result*, orally or mentally, and a little practice enables the computer to do so.

Multiplication.

29. Multiply each digit in any column of Table 2 by one and the same number, and add the results. The sum is 10 times the product in the line derived from 5.

Ex. Multiply each digit in col. 2 by 3, the results are 6, 24, 18, 12, 15, 3, 9, 21, 15, 27, the sum of which is 150, or 10 times 15, the line resulting from 5; or, more readily, the result is 50, multiplied by the common multiplier; 3 times 50 being 150.

30. Multiply each digit in any column of Table 2 by one and the same number, and increase the products by some other constant number. Add the results, and the sum will be 10 times the result derived from the line containing 5.

Ex. Multiply each line in col. 2 by 4, and increase each result by 2 in one operation. The results are 10, (*i. e.* 4 times 2, and 2) 34, 26, 18, 22, 6, 14, 30, 22, 38, the sum of which is 220, or 10 times 22 the line derived from 5. These results should be caught and written down at once, *not* as the result of *two* operations.

31. Multiply each number in any column of Table 2 by one and the same number, and increase the result by the corresponding digit in any other column. Add the results, and the sum is 10 times the result derived from multiplying 5 by the same multiplier, and increasing this product by 5.

Ex. Multiply each line in col. 2 by 4, and increase each result by the corresponding number in col. 3. The results are 11, 37, 33, 22, 25, 11, 13, 32, 22, 44, the sum of which is 250 or 10 times 25, which is 4 times 5 increased by 5.

32. Multiply each number in any column of Table 2 by one and the same number, and add the corresponding number in any other column in one operation, and then increase the result by the corresponding number in any third column in a second operation. The sum of the results is 10 times the result derived from multiplying 5 by the constant multiplier, and increasing this product by 10.

Ex. Take the three first columns of Table 2, and 3 as the constant multiplier. From the first lines form 3 times 1 and 2, are 5, and 3 are 8; only say 5, 8, and write down 8 only. From the second lines form 3 times 3 and 8 are 17, and 5 are 22; only say 17, 22, and write down 22 only; and so on. The results are 8, 22, 30, 31, 37, 14, 16, 29, 31, 32, and their sum is 250 or 10 times 25, 25 being 3 times 5 increased by 10.

The pasteboard slips will be found very convenient for these examples, as they may be readily combined in twos and threes, producing an almost endless variety. By using any two slips and requiring the products of adjacent numbers to be named at sight, the Teacher will most conveniently exercise the pupil in the multiplication table. This exercise is also much more usefully performed by pointing to the numbers than by calling them out, as in actual practice it is the eye, and not the ear—an image, and not a sound—which has to be employed.

Short Division.

33. Consider the corresponding numbers in two adjacent columns of Table 2, as forming the tens and units of the same number. Divide each by the same

digit, and write down the quotients and remainders. Add the remainders, divide the result by the constant divisor, put down the remainder, and carry the quotient to the column of quotients, which add. The results will be the quotient and remainder, found on dividing 5 5 0 by the constant divisor.

Ex. Reading the corresponding numbers in cols. 1 & 2 in Table 2 as the single numbers, 1 2, 3 8, 5 6, &c., and using 7 as the constant divisor, the results are—

Dividends ...	1 2,	3 8,	5 6,	7 4,	9 5,	2 1,	4 3,	6 7,	8 5,	5 9	...	5 5 0
Quotients.....	1,	5,	8,	1 0,	1 3,	3,	6,	9,	1 2,	8	...	7 8
Remainders...	5,	3,	0,	4,	4,	0,	1,	4,	1,	3	...	4

Sum *of* first remainders, 2 5, which divided by 7 gives a quotient 3, and a remainder 4, the same as the last remainder. Adding this quotient 3 to the sum of the first quotients, the result is 7 8, the last quotient. Only the two last lines are written in practice.

As the principal effect of this exercise is to enable the pupil to catch the difference between a product and a given number, the *remainders* are the important numbers to preserve. If then we only write the *last* line (instead of the *two* last), we may consider it a sufficient proof of correctness that on dividing both the sum of these remainders and 550 by the common divisor, the two remainders are the same; in the present case, they are both 4.

Long Division.

Preparation for performing the multiplication and subtraction in one line.

34. Reading the two corresponding numbers in any adjacent columns as the tens and units of the same number, subtract them from the corresponding number in the next column, borrowing the requisite number of tens. Write down the differences and the borrowed tens in separate lines. Add each. The sum of the first line increased by 5 0 0 will be the sum of the second.

Ex. Reading the corresponding numbers in the first and second columns of Table 2 as single numbers, 1 2, 3 8, 5 6, &c., subtract

12 from 3, borrowing 10, 38 from 5, borrowing 40, 56 from 9, borrowing 60, and so on. The results are as follows—

Minuends ... 3, 5, 9, 6, 5, 7, 1, 4, 2, 8
Subtrahends... 12, 38, 56, 74, 95, 21, 43, 67, 85, 59
Differences ... 1, 7, 3, 2, 0, 6, 8, 7, 7, 9 sum 50
Borrowed..... 10, 40, 50, 70, 90, 20, 50, 70, 90, 60 sum 550

The last sum exceeding the first by 500.

Only the two last lines are written in practice, and the zeroes may be conveniently omitted, as one tenth the sum borrowed will exceed one tenth the sum of the differences by 50.

35. Multiply each digit in any column in Table 2 by the same number, increasing the result by the corresponding number in the next column in one operation (as in **31**), and subtract this result from the corresponding number in the next adjacent column, borrowing the necessary number of tens. Write down the differences, and the borrowed tens in separate lines. Add each line. The sum of the first line increased by 50 times the common multiplier gives the sum of the second line.

Ex. Take 3 as the common multiplier, multiply the first column, adding in the second, and subtracting the result from the third. Form thus 3 times 1 increased by 2 or 5, which taken from 3 gives remainder 8 with 10 borrowed; say 5 and 8, 13, write 8, 10. Form 3 times 3 increased by 8 or 17, which taken from 5 gives 8 with 20 borrowed; say 17 and 8, 25; write 8, 20; and so on.

Form Differences ... 8, 8, 8, 1, 3, 0, 6, 9, 3, 4
Borrowed 10, 20, 20, 20, 30, 0, 20, 30, 30, 20

The sum of the first line increased by 3 times 50 or 150 is 200, which is the sum of the second line. Or, omitting the zeroes, one tenth of the first line increased by 15 (or 3 times 5) is 20, which is the sum of the second line omitting final zeroes. See the application of this exercise in (**63**).

The pasteboard slips will be exceedingly useful in working these exercises.

36.—AUXILIARY TABLE.

Line.	A. Number.	B. Complement.
1	123 654 854 168 7	876 345 145 831 3
2	127 643 417 875 3	872 356 582 124 7
3	135 638 288 072 2	864 361 711 927 8
4	178 929 422 875 8	821 070 577 124 2
5	218 493 534 167 6	781 506 465 832 4
6	243 204 324 488 3	756 795 675 511 7
7	257 623 674 147 4	742 376 325 852 6
8	267 961 876 623 2	732 038 123 376 8
9	316 517 393 375 1	683 482 606 624 9
10	324 384 314 311 5	675 615 685 688 5
11	348 467 799 653 7	651 532 200 346 3
12	368 293 445 632 3	631 706 554 367 7
13	425 768 164 667 4	574 231 835 332 6
14	459 384 837 696 8	540 615 162 303 2
15	478 395 248 661 2	521 604 751 338 8
16	487 629 237 311 9	512 370 762 688 1
17	536 773 144 383 7	463 226 855 616 3
18	546 540 875 776 5	453 459 124 223 5
19	563 764 683 581 3	436 235 316 418 7
20	578 425 970 368 2	421 574 029 631 8
21	642 512 242 189 6	357 487 757 810 4
22	653 294 184 733 8	346 705 815 266 2
23	677 761 634 304 5	322 238 365 695 5
24	685 429 074 513 8	314 570 925 486 2
25	727 057 727 488 4	272 942 272 511 6
26	760 548 465 152 6	239 451 534 847 4
27	777 926 488 373 3	222 073 511 626 7
28	786 841 735 168 5	213 158 264 831 5
29	847 396 215 847 9	152 603 784 152 1
30	863 617 338 667 6	136 382 661 332 4
31	875 643 487 263 4	124 356 512 736 6
32	888 889 683 375 2	111 110 316 624 8

On the Use of the Auxiliary Table.

37. The preceding Table is intended to facilitate the formation of subcomplements, complements, binōnals, and nōnundecimals; but as these can be formed with perfect facility by the methods already explained (see *Definitions*, &c.), the Table is merely an *auxiliary*, and not an indispensable requisite, and hence its name. (*App.* **21.**)

38. *Subcomplement.* Select any number, or portion of a number from column A, excluding the last right hand digit. Select the corresponding digits from column B. The second number is the subcomplement of the first.

Ex. Take the first six digits in column A, line 1, viz. 1 2 3 6 5 4. The corresponding digits in column B, line 1, are 8 7 6 3 4 5, and this number is the subcomplement of the former, see (**1.**)

39. *Complement.* After the number selected (as in **38**), write the last digit of any line in A, and after its subcomplement write the corresponding digit from B, or add 1 to the subcomplement.. The numbers will now be complementary (**6**).

Ex. Having found 1 2 3 6 5 4 and its subcomplement 8 7 6 3 4 5, as in the last example, write after them 6 and 4, the last digits of A 1, or 3 and 7, the last digits of A 27, &c., or add 1 to 8 7 6 3 4 5, the subcomplement. The results, 1 2 3 6 5 4 6 and 8 7 6 3 4 5 4, or 1 2 3 6 5 4 3 and 8 7 6 3 4 5 7, or 1 2 3 6 5 4 and 8 7 6 3 4 6 are complementary. Any complete lines in A and B, or the same number of final digits in corresponding lines of A and B, are complementary.

40. Binōnals and nōnundecimals in endless variety may now be immediately formed by the help of the subcomplements, inserting the requisite number of nines (see **2** and **16**).

Ex. The number 1 2 3 6 5 4, followed by its subcomplement 8 7 6 3 4 5, found as in (**38**), produces the binōnal 1 2 3 6 5 4 8 7 6 3 4 5, which is also nōnundecimal, because its *second* part (8 7 6 3 4 5) consists of an *even* number of digits (**16**). Now 1 2 3 followed by its subcomplement 8 7 6—that is, 1 2 3 8 7 6—is also binōnal. It is, however, not nōnundecimal, because its extent (**4**) is uneven. Insert a 9 between its two parts, and the result 1 2 3 9 8 7 6 is both binōnal and nōnundecimal (**16.**)

ADDITION.

I. — *Examples for Self-Practice.*

41. Add all the digits in any corresponding numbers in columns A and B of the Auxiliary Table (36) backwards or forwards, the result will be 118. (*App.* 22.)

Ex. Line 1. The sum of 1, 2, 3, 6, 5, 4, 8, 5, 4, 1, 6, 8, 7, 8, 7, 6, 3, 4, 5, 1, 4, 5, 8, 3, 1, 3 is 118.

42. Add any number of digits in any numbers in A (excepting the last right hand digits in those numbers) to the corresponding digits in B, and to the *number* of the digits selected from A, the result will be ten times this last number. (*App.* 23.)

Ex. If you add 6 digits in A (excepting as above) to the corresponding digits in B and to 6 (the *number* of the digits taken from A), the result will be 60. Thus take the first 6 digits in line 1 ; the sum of 1, 2, 3, 6, 5, 4 and 8, 7, 6, 3, 4, 5 and 6 is 60.

Or take the digits in six lines of the second vertical column in A, and add to those in B and to 6, the result will be 60, as before, for 2, 2, 3, 7, 1, 4 added to 7, 7, 6, 2, 8, 5 and to 6 are 60.

43. These two examples, sedulously practised for half an hour a day, will make a rapid and certain adder. As the difficulties of addition increase according to the number of digits added in succession, and not at all according to the number of rows of digits added, the practiser can increase his difficulties at pleasure. Thus if he take 32 digits, or one vertical row or column in A, and add to the corresponding vertical row in B and to 32, the result is 320. Or if he add all the digits in the first 12 vertical rows in A, as if they formed a single row of 384 digits to the corresponding 384 digits in B and to 384, the result of this addition will be 3840, and if any practiser obtain this result rapidly and correctly, he may consider himself perfect in addition.

II. — *Examples for School or Family Use.*

44. In school practice, or in private teaching, it is necessary that the pupil should not be able to guess the result beforehand, as idle pupils might otherwise content themselves with writing down the result, without performing the operation. It is also necessary that the Teacher should waste the least possible time in setting examples. These two objects are attained as follows.

45. Write on the black board any number whatever, followed by a series of numbers and their complements (which may be taken from the *Auxiliary Table* 39), each containing the same number of digits.

Thus—any number.....	978	632	452
Numbers from col. A.	218	493	534
	368	293	445
	487	629	237
Their complements from col. B.	781	506	466
	631	706	555
	512	370	763

Make each pupil copy this upon his slate.

Each pupil will now supply a number for himself, of the same number of digits, and write under the above. He may, if he chooses, write down any one of the numbers before him as the new number. Suppose one pupil selects 654 283 721. He has then to add together—

978	632	452
218	493	534
368	293	445
487	629	237
781	506	466
631	706	555
512	370	763
654	283	721
Sum... 4632	916	173

Here the sum of all the *eight* lines is the sum of the *first and last lines* increased in the tenth or extra left hand place by 3, the number of the numbers taken from col. A. That is, as the sum of the first and last lines being 1632916173, the complete sum is 4632916173, which the master can check at a glance, and without betraying the secret to the pupil. (*App.* 24.)

46. The following is a simple, and, in practice, most convenient variation of the preceding process, consisting in adding the initial number to the last complement, or subtracting it from the same, prior to writing the numbers on the black board.

Assume any easily remembered number, as a series of ones, twos, threes, alternate ones and twos, &c., and bear it in mind as the "proof number."

Write down on the black board any numbers from column A, and the complements of all but one from column B, as in (45). In lieu of this omitted complement, write the sum arising from adding it to the proof number, or the difference arising from diminishing it by the proof number.

Let each pupil copy these numbers, and supply a last number for himself at pleasure (not 1 1 1 ... or 1 2 3 ...).

The sum may be read at sight by adding the proof number to the number supplied by the pupil, or subtracting it from the same and prefixing the number of the numbers taken from A. (*App.* **25.**)

Ex. In these examples the lines marked *a*, *b*, *c*, are the numbers from column A. The lines *a'*, *b'*, are their complements from B; *p* is the proof number; *P* the proof number, with 3, the number of the numbers taken from column A, prefixed; *C* is a complement of *c*, with the proof number added to it, or subtracted from it; *n* the number supplied by the pupil; and *N* the same number with 3, the number of the numbers from column A, prefixed.

	Ex. 1.	*Ex.* 2.	*Ex.* 3.
a	5367	1236547	1236547
b	7643	2679612	2679612
c	4252	3682933	3682933
a'	4633	8763453	8763453
b'	2357	7320388	7320388
C	6859	8539289	11872622
n	293	2583112	2583112
Sum	31404	34805334	38138667
being the sum of *n*	293	2583112	2583112
and *P*	31111	32222222	35555555

	Ex. 4.	*Ex.* 5.	*Ex.* 6.
a	5367	1236547	1236547
b	7643	2679612	2679612
c	4252	3682933	3682933
a'	4633	8763453	8763453
b'	2357	7320388	7320388
C	4637	5205956	4094845
n	293	2583112	2583112
Sum	29182	31472001	30360890
being *N*	30293	32583112	32583112
less *p*	1111	1111111	2222222

The Teacher will discover the correctness of the sum in all cases by *addition;* in *Ex.* 1, 2, 3, by adding p to n, and prefixing the proper number to produce the sum; in *Ex.* 4, 5, 6, by adding p to the *sum*, and *rejecting* the proper prefix to produce n.

When the auxiliary table is not used, C is found by subtracting c in *Ex.* 1, from 11111, in *Ex.* 2, 1222222, in *Ex.* 3, from 1555555; that is, in each case from a number formed by prefixing 1 to p; in *Ex.* 4, from 8889, in *Ex.* 5, from 888889, in *Ex.* 6, from 777778—that is, in each case from the complement of p, a number which, when added to p, produces 1 followed by zeroes.

By varying these forms, the acutest pupil may be completely baffled in guessing the law of the result.

47. *To set a new example*, the teacher has only to make the pupil rub out his last line which he wrote for himself, replace it by another, and add again. If it is suspected that the pupil adds up the constant lines on the board once for all, and only adds this *result* to the new line he writes, the teacher must set him a new example each time, and this is managed either by making him select different pairs of complements from the black board each time, or as follows (**48**).

48. In the above process every digit is checked with certainty, but the fact of the whole result being correct may be discovered by using nonundecimals, by which all attempts at *forcing* a result will be frustrated.

Write a succession of nonundecimals on the board, and bid the pupils select any 6 or 7 or more of them, and add,

each pupil selecting different numbers. The sums will be nonundecimal (*App.* 11); and to set a new example, any one of the old lines may be rubbed out, the old total used as a new last line, and a new addition commenced.

Ex. Take the nonundecimals from the auxiliary Table (40.)

```
                12368763
                 2189781
4            18493981506
                    396
              -----------
Sum          18508540446
```

which is nonundecimal (17.)

Now omit 396, and add in the sum to the other numbers, the result is 37017080496

which is also nonundecimal.

Omitting the second line, and adding in the total, we obtain 27015891111

which is also nonundecimal.

☞ 49. As the time required for setting examples is now reduced to such a small amount, the Teacher should never commit the error of making a pupil who has obtained an erroneous result, labour over an example till it is right. The pupil's error has probably arisen in only one or two combinations at most out of a great many, the rest having been rightly performed. If compelled to re-work the example, he will most probably fix the mistakes in his mind by constantly reckoning these combinations incorrectly. The proper mode to pursue then is simply to rub out the result, and set another example. But here again the pupil should not be forced to do a fixed number of examples correctly. As soon as the pupil's head becomes at all heavy over calculation, the master who forces him to proceed only defeats his own object, and does harm to the child's brain. Care should be taken therefore, at first especially, to make the examples easy, by using only a very few lines, beginning with 4 or 5, and seldom extending beyond 10 or 12, till the pupil is very ready at adding.

If possible, do not proceed to subtraction till the pupil can add up a column of 20 figures with perfect correctness and great rapidity.

50. Instead of writing lines out on the black board for the class to copy, the Teacher may often read them out in words at length, and make the pupils write them in figures. This exercise in numeration is very important.

SUBTRACTION.

I. — *Examples for Self-Practice.*

51. Take 10 times any number, and subtract from it the same number of digits (excluding last right hand digits) in col. A, and the corresponding digits in col. B, the last remainder will be the original number. (*App.* **26.**)

Ex. Take 6, and from 10 times 6 or 60 subtract 6 digits in A, as 1, 2, 3, 6, 5, 4, and the corresponding 6 digits in B, viz. 8, 7, 6, 3, 4, 5. The successive remainders are 59, 57, 54, 48, 43, 39, 31, 24, 18, 15, 11, 6, the last being 6, the original number.

52. Subtract any line or part of a line in A from any other, and subtract the corresponding numbers in B, borrowing a 10 in the subtracting of the last digits on the left where necessary. The results will be *complementary*. (*App.* **27.**)

Ex. Take the last 7 figures in the first two lines.

8541687	(1)1458313
4178753	5821247
4362934	5637066

In the last subtraction 10 was borrowed, indicated by the prefixed (1). These results are complementary, as is readily seen.

53. If all the digits in a whole line in both A and B be considered as one number, and be subtracted from a larger number formed, in the same way, the result is *binonal*. (*App.* **28.**)

Ex. Subtract the first from the fifth line,

2184935341676, 781506465832 4
1236548541687, 876345145831 3

9483867999 88, 905161.3200011

where the comma marks the end of the number taken from A.

These examples will, of course, be performed from the table, without writing down anything but the result, and this result, *increased by* 1, may be subtracted from a similar number, and so on for ever.

II. — *Examples for School or Family Use.*

54. Single examples in subtraction are immediately verifiable by addition.

Write on the board a series of binonals of the *same extent*, and bid the pupils subtract any one of them from a greater, and the remainder from another, and so on. All the results will be binonal. (*App.* **29.**)

This will also be good practice in subtracting with the larger number uppermost, which should be as familiar as the ordinary routine.

Ex. Write on the black board the binonals

a	123654, 876345
b	218493, 781506
c	768164, 231835
d	536773, 463226

Subtract the first from the second.

from *b*	218493, 781506
take *a*	123654, 876345
take	94838, 905161
from *c*	768164, 231835
from	673325, 326674
take *d*	536773, 463226
	136551, 863448

which we might take from *c*, and so on for as many lines as we pleased. All the results are binonal, as is readily seen by the position of the commas, which are of course not written in actual practice.

In this way examples of subtraction may be made as few or many as we please, the verification for the last verifying all. But this verification only points out *one* of *two* digits as erroneous, and not the precise digit. The following process gives the *precise* digit.

55. Write on the board a series of numbers from A, all having the same number of digits, and the complements of all but the last, from B. Above them and to the left write a digit larger by 1 than the number of complements. Bid the pupil fill up the deficient places in this number at pleasure, and subtract all the other numbers from it in order.

The last difference added to the middle number on the board will make up the original minuend less the number of complements in the last place to the left, as will be more clearly shown in the examples. (*App.* **30.**)

New examples are set by altering the last figures of the minuend, always preserving the single digit written on the board.

Ex. 1. Take only two numbers, and the complement of the first. Write on the board

2
1687
8573
8313

Here 2 is written above because there is only one complement, namely 8313, the complement of 1687. The pupil has to supply 4 digits in the minuend at pleasure, to make the top line reach as far to the right as the others. He supplies, suppose, 9365, and then the work proceeds thus—

29365
1687
27678
8573
19105
8313
10792

Here 10792, the last difference, added to 8573, the middle number, gives 19365, the minuend less the 1 or the number of complements in the last left hand place.

To set a new example we have only to change the digits, 9365, into any others, as 5281, and then use 25281 as a new minuend, from which to subtract the other three numbers successively. The differences will now be 23594, 15021, and 6708, and this last difference added to the middle number, 8573, gives 15281.

Ex. 2. Take 4 numbers and 3 complements. The numbers written on the board, and the example as worked, will be as follows.

Suppose 3 2 5 3 to be the digits supplied after the 4, so that 4 3 2 5 3 becomes the minuend.

4

2 8 5 7	4 3 2 5 3	original minuend.
3 6 2 4	2 8 5 7	
5 9 8 3		
2 3 7 8	4 0 3 9 6	
7 1 4 3	3 6 2 4	
6 3 7 6		
4 0 1 7	3 6 7 7 2	
	5 9 8 3	
	3 0 7 8 9	
	2 3 7 8	middle number.
	2 8 4 1 1	
	7 1 4 3	
	2 1 2 6 8	
	6 3 7 6	
	1 4 8 9 2	
	4 0 1 7	
	1 0 8 7 5	last difference.

Here the last difference, 1 0 8 7 5, added to the middle number, 2 3 7 8, gives 1 3 2 5 3, the original minuend diminished by 3 (the number of complements) in the last left hand place.

Examples of the first kind are long enough in school practice, except as an exercise to very ready computers.

MULTIPLICATION.

I. *Examples for Self-Practice.*

56. Multiply any line in A and the corresponding line in B by any number whatever. The sum of the products will be the multiplier followed by as many zeroes as there are digits in either multiplicand. (*App.* **31.**)

Ex. 1. The products of the first line in A and B by 8 are respectively 989 238 833 349 6

And 7010 761 166 650 4

which when added produce 8, followed by 13 zeroes.

The learner will perform these exercises direct from the table, only writing down the results without copying out the multiplicands. It may be convenient to write the complementary numbers under one another, as above; but in that case it will be best to conceal the first product while the second is being formed.

A variety of other exercises may be formed, but they are quite unnecessary; steady practice of this one method will suffice to give rapidity and certainty in multiplication.

The learner has no occasion to confine himself to a single digit as the multiplier; any number, however large, will answer equally well, but when the multiplier is large, it will be more convenient to reduce the number of digits in the multiplicand by omitting some of those on the *left* hand, so as to leave complementary numbers for the multiplicands. The learner can therefore exercise himself in multiplying by 12, 13, 14, 15, 16, 17, 18 and 19 in one line, &c.

Ex. 2.

8541687	1458313
253	253
25625061	4374939
42708435	7291565
17083374	2916626
2161046811	368953189

The same proof applies to each partial product.

Proof 2161046811

2530000000

There will of course be no necessity for writing the numbers under each other for the proof, as in the example. The self-practiser or teacher will begin with the right hand digits of each product, and see that the sum of the two last digits on the right is 10, of the two next 9, and continually 9, until he arrives at the digits of the multiplier diminished by 1.

II. *Examples for School or Family Use.*

57. Write on the board a number of binonals, each containing one middle nine (2). Let the pupil select his number and his multiplier, the latter being of one digit. The result is binonal. (*App.* 4.)

Ex. Take the binonal 1239876 from line 1 of table.
Multiply by 6

Product 7439256 which is binonal.

As this binonal happens to have a middle nine, it may itself be used as a new multiplicand, thus—

7439256
7

52074792 which is binonal.

By inserting a middle 9 this result may be used as a new multiplicand, thus—

520794792
8

4166358336 which is binonal.

58. Write on the board a series of numbers separated from their complements by a zero, thus—

168708313
875301247
722092778
875801242

Let the pupil select his number and multiplier. The products will separate into two portions, which are clearly distinguished by the zero in the multiplicand, and their sum will be the multiplier followed by zeroes. (*App.* 32.)

Ex. 168708313
8

1349666504

where the result separates into

13496 as far as under the 0 in the multiplicand.
And 66504 beginning from under the 0 in the multiplicand.
Sum 80000

This example will serve to baffle a pupil who has an inkling of the law of binonals. And this purpose may be more effectually carried out by writing some binonals among these numbers on the black board for the pupil's selection, thus—

1239876
756202438
25699743
9130087

The Teacher sees at once by the middle 0 or 9, which kind of example he is dealing with, and can check the result accordingly, but as the two laws are very different, they will be more difficult for the pupil to discover.

59. For multipliers exceeding one digit, use binonals with as many middle nines as there are digits in the multiplier (*App.* 4); or else use bicomplementary numbers (as in 58) with as many middle zeroes as there are digits in the multiplier. The verification in this case is precisely similar to that in (57) for binonals, and that in (58) for bicomplementaries.

Ex. 12399876
78

99199008
86799132

967190328
which is binonal.

1687008313
63

5061024939
10122049878

106281523719
106281

630000

But the introduction of so many middle nines or zeroes renders the example so clearly artificial, that in the case of long multiplication it is preferable, either to split each example into two, as in (56), or to use nonundecimals as in (60.)

60. Write down a few nonundecimals (**16**). Let the pupil select his multiplicand and multiplier, the number of digits in the latter being assigned, or a few multipliers being written for his selection. The product is nonundecimal (*App.* **15**), as may be proved by last sums and difference (**11** and **14**); and it may, therefore, be used as a new multiplicand.

Ex. Write down on the black board the following nonundecimals

1239876
49355064
8316

Then the pupil selects, say 49355064 as a multiplicand
and supplies any other number 347 as a multiplier

345485448
197420256
148065192

Product 17126207208

which is nonundecimal. It is totally impossible for the pupil to force such a result, as a knowledge of the proof is no guide to the figures in the result.

Taking this product as a new multiplicand, and multiplying it by any number, as 589, we have

17126207208
589

154135864872
137009657664
85631036040

10087336045512

which is also nonundecimal. If a series of such examples are worked, the product being always used as a new multiplier, it will be sufficient to verify the last result, the correctness of which necessarily implies that of all the preceding, because they have been all employed in producing it, and hence this is a very convenient form for self-practice.

DIVISION.

I. *Examples for Self-Practice.*

61. Select any divisor (beginning of course with those consisting of a single digit, as 2, 3, 4, &c., for the practice of short division), and divide by it any line or portion of a line in A, not containing its last right hand digit. Prefix a number which is one less than the divisor, to the corresponding digits in B, and divide the result. The quotients will be *sub*complementary. (1.) There will always be one or two remainders, and their sum will be the divisor less 1. (*App.* 33.)

For a fresh example: if the two quotients have the same number of digits, prefix the remainders to the quotients for new dividends, and divide again by the former divisor. The new quotients and remainders will follow the same law. And so on for ever.

If one of the quotients has one more digit than the other, remove its first digit, which will be a 9, before prefixing the remainders to the quotients, and proceed as before.

Ex. Divisor 7; select 123654, the first six digits of line 1 in A as the first divisor, and prefix 6 to 876345, the corresponding digits in B, to form the second divisor.

Then 7)123654 and 7)6876345

a... 17664.6 *b*... 982335

Here there is only one remainder, and the second quotient has one more digit than the first. Hence rejecting its initial 9, and putting the 6 before it thus reduced or before the first quotient, we have as new examples, either:—

7)617664 from *a* and 7)82335 from *b*

c... 88237.5 *d*... 11762.1

Or 7)17664 from *a* and 7)682335 from *b*

e... 2522 *f*... 97477.6

From the above quotients and remainders we can form fresh examples in precisely the same way as follows:—

7)588237	from *c*	and	7)111762	from *d*
g... 84033.6			*h*... 15966	
Or 7)188237	from *c*	and	7)511762	from *d*
26891			73108.6	
And 7)62522	from *e*	and	7)7477	from *f*
8931.5			1068.1	
Or 7)2522	from *e*	and	7)67477	from *f*
360.2			9639.4	

and so on, in all of which we observe the rule to hold.

62. The same plan answers for long division. But we may proceed more conveniently by selecting corresponding digits (excluding the last on the right) from A and B, and after prefixing any numbers to them to form the dividends, using *one more* than the sum of the prefixed numbers as the divisor. (*App.* **33.**)

Ex. Suppose 371 and 246 are the prefixed numbers. Then as their sum is 617, we must take 618 as the divisor. Let the digits selected from the *Auxiliary Table* be the first six in line 1.

600523 quotient	399476 quotient
618)371123654	618)246876345
3708	1854
3236	6147
3090	5562
1465	5856
1236	5562
2294	2943
1854	2472
440	4714
	4326
	3885
	3708
	177

Here the sum of the remainders 440 and 177 is 617, or one less than the divisor, while the quotients are subcomplementary.

The learner will of course conceal the first quotient while he is obtaining the second, to prevent suggestion.

New examples may be formed as before by prefixing the remainders, 440 and 177, to the quotients, and thus obtaining the new dividends, 440600523 and 177399476, or 440399476 and 177600523, using the same divisor, 618. The law of the quotients and remainders holds as before.

In working this example the quotient is written *above* the dividend, so that each digit of the quotient is placed over the corresponding digit, which is *brought down* from the dividend—a plan much recommended as marking distinctly what digits are brought down, and as very useful for determining the position of the point in the division of decimals. If preferred the quotient may be written *under* the dividend as in short division, in which case the first subtraction will be written one line lower, and the first partial dividend inserted, thus—

```
618)371123654        This method is more convenient
    ---------        when the dividend has been ob-
     600513          tained as the result of another
  3711               operation, as by a multiplication,
  3708               &c. If this quotient has to be
  -----              divided again, the work now
    3236             written beneath it may be written
    3090  &c.        on another slip of paper, the re-
    ----             mainder being duly preserved.
```

63. The learner who has carefully performed the preliminary exercises (**34.**) and (**35.**) will be able to multiply the divisor by each digit in the quotient, and subtract this product from the partial dividend in one operation, and the results of his division of the two last examples will now appear in the following form.

```
        600523 quotient          399476 quotient
618)371123654            618)246876345
    3236                     6147
     1465                    5856
      2294                    2943
       440                     4714
                                3885
                                 177
```

Here the results are obtained thus:—3711 is the first partial dividend, from which we have to take 6 times 618. Then 6 times 8 are 48, and 3 are 51,—say 48 and 3, 51, write down the 3 and carry the 5. Next 6 times 1 are 6 and 5 are 11, 11 and 0 are 11,—say 11 and 0, 11, put down 0 and carry 1. Next, 6 times 6 are 36 and 1 are 37, 37 and 0 are 37,—say 37 and 0, 37, and put

down 0. The whole process is repeated thus, accented figures being written down and figures preceding a line / being carried.

48 and 3′, 5/1 ; 11 and 0′, 1/1 ; 37 and 0′, 37.

The second partial quotient is 3236, and the new digit is the quotient 5 ; we have therefore to take 5 times 618 from 3236 by a similar process, which will be repeated thus—

40 and 6′, 4/6 ; 9 and 4′, 1/3 ; 31 and 1′, 32.

Again, to take 2 times 618 from 1465, we have

16 and 9′, 2/5 ; 4 and 2′, 6 ; 12 and 2′, 14.

And to take 3 times 618 from 2294 we have

24 and 0′, 2/4 ; 5 and 4′, 9 ; 18 and 4′, 22.

After a little careful practice, this mode will be found as secure as the old, and much more rapid, saving many figures.

64. The simplest mode, however, of forming examples in long division is the following.

Divide any nonundecimal by any other nonundecimal, adding zeroes at pleasure to the dividend. All the remainders from the bringing down of the first zero will be nonundecimal (*App.* **14**); and if the divisor do not exceed 4 digits, they will be also binonal (**17** and *App.* **17**).

Ex.

```
                378596        or as in (63)  378596
3267)1236876300                3267)1236876300
        9801                           25677
        ------                          28086
        25677                           19503
        22867                          *31680
        -----                            22770
         28086                          . 3168
         26136
         -----
          19503
          16335
          -----
         *31680
          29403
          -----
            22770
            19602
            -----
             3168
```

From * all the remainders are binonal, and as the last remainder is the same as the first binonal remainder, it is useless proceeding, for the same remainders will necessarily continually recur.

This process leaves the figures of the quotient unverified, but when the learner wishes he may always verify them thus.

Taking the last remainder as a new divisor, and the old divisor as a new dividend, divide out as far as possible. Then treat the new remainder in the same way, and so on till you come to a division without a remainder, which will always be the case.

By the divisor in this last division (which will most frequently be 99) divide the original divisor, original dividend, and original last remainder. All three will divide out without remainders.

Use the first of these three quotients as a new divisor, and the second as a new dividend.

The quotient will be the same as the original quotient, and the remainder will be the third of the three quotients mentioned above. (*App.* 34.)

Thus, continuing the above example, where the remainder is 3168, and the original divisor 3267, we have as in the margin.

```
          1
3168)3267
     3168
     ----      32
       99)3168
          297
          ---
          198
          198
          ---
```

Dividing the original divisor, 3267, the original dividend, 1236876300, and original remainder, 3168, by 99, the last divisor, we obtain the three quotients 33, 12493700 and 32. And dividing the second of these by the first, we have as in the margin, where the remainder, 32, is the third quotient found above, and the quotient is the original quotient in the example on the last page.

```
      378596
33)12493700
   99
   --
   259
   231
   ---
    283
    264
    ---
     197
     165
     ---
      320
      297
      ---
       230
       198
       ---
        32
```

II. *Examples for School or Family Use.*

A. *Short Division.*

65. Write down any pairs of *sub*complementary numbers from A and B.

Write down the divisor, and under it pairs of numbers whose sum amounts to the divisor less 1.

Bid the pupil select his pair of large numbers, prefix to one of them one number from a small pair, and to the other the other, and divide them by the divisor proposed.

The quotients will be subcomplementary, and the sum of the remainders will be the divisor less 1.

For a new example, proceed as in **(61)**.

Ex. For divisor 7, write on the board

7		123654	876345
0	6	854168	145831
1	5	218493	781506
2	4	534167	465832
3	3	683482	316517
4	2	Then the pupil selects one pair of dividends (*e.g.*,	
5	1	the last 683482 and 316517) and one pair of	
6	0	prefixes (as 5 and 1) and thus forms the examples.	

7)5683482 and 7)1316517

a... 811926 *b*... 188073.6

from which he may form the new examples :—

7)811926 from *a* and 7)6188073 from *b*

c... 115989.3 *d*... 884010.3

And then 7)3115989 from *c* and 7)3884010 from *d*

445141.2 554858.4

and so on for ever.

It is clear that if a series of such examples be formed the verification of the last pair is a sufficient verification of all.

66. If pupils detect the law of the relation of the quotients in **(65)**, it will be necessary for the teacher to baffle them. This is very simple. Having selected the

divisor and formed the numbers on the board, as directed in (**65**), *before* the meeting of the class, *increase or diminish by the divisor* one or more known places of digits in the column taken from B. Then, in the quotient that place must exceed or fall short by 1 of the digit which should have been there according to the common rule of (**65**).

Thus taking the examples selected in (**65**), and supposing the third place from the right to be augmented by 7, the examples will become

$$7)5683482 \qquad \text{and} \qquad 7)13172\overset{*}{1}\overset{*}{7}$$

$$811926^{*} \qquad\qquad 188173.6^{*}$$

The stars point out the changes. The black board after these changes in the *Ex.* in (**65**) will present the appearance

7		123654	8$\overset{*}{7}\overset{*}{7}$045
0	6	854168	146531
1	5	218493	782206
2	4	534167	466532
3	3	683482	317217
4	2		
5	1	Two digits are affected by this addition of	
6	0	7, as shown by the stars.	

When this is done, however, fresh examples cannot be formed from the quotients as in (**61**), but all examples must be taken directly from the board.

B. *Long Division.*

67. Write out on the board two series of nonundecimals, one of them much smaller than the other.

Divide any larger nonundecimal by any smaller one; the remainder will be nonundecimal.

For a fresh example, use the old or any new nonundecimal dividend, and the remainder as a new divisor, and so on for ever.

If the divisor have only 4 digits, the remainders will also have no more than 4 digits, and will be, therefore, binonal (**17** and *App.* **17**).

Thus write on the board

Divisors.	Dividends.
3465	654854345145
2871	417875582124
5346	623674147937632585 2
6237	267961732038

This alone furnishes 16 examples, and each example furnishes a new example by giving a new divisor from its own remainder, when there is one. If there is no remainder, as is possible, this must also be considered a sufficient proof of correctness. Thus on dividing the first of the above dividends by the first divisor, the quotient is 188991153, and there is no remainder.

SELF-PRACTICE IN THE FOUR SIMPLE RULES.

68. Multiply any nonundecimal by any multiplier (which need not be nonundecimal), and divide the result by a nonundecimal; the remainder will be nonundecimal, and this will be sufficient proof of correctness. If in the division the multiplication and subtraction be performed at once (as in **63**), these examples will exercise the learner in all the processes. The learner should endeavour to attain great rapidity as well as correctness.

Using this remainder as a multiplier, and multiplying any number by it, the result is again nonundecimal, and dividing this by a nonundecimal divisor, the learner will obtain a nonundecimal remainder, and so on. Verification of the last result is sufficient verification for the whole process.

Ex. 2574 and 3762 being nonundecimal, multiply any number, as 36507862, by one and divide the result by the other, the remainder will be nonundecimal, that is, in this case, the divisor having only 4 digits, binonal.

COMPOUND RULES.

69. Although the contemplated introduction of a decimal coinage, and consequently of a decimal system of weights and measures, will hereafter render the study of compound rules unnecessary, no system of examples would be now complete which did not embrace them. Now, it happens that the above rules for simple arithmetic apply, so far as complements and nonundecimals are concerned, almost equally well to compound rules. We have only first to define what we mean by compound complements and supercomplements, and compound nonundecimals.

70. *Compound Complement.* If we subtract 365*l.* 15*s.* 7¾*d.* from 1000*l.*, we obtain 634*l.* 4*s.* 4¼*d.* This result may be obtained immediately by subtracting each digit of the highest denomination from 9, and then each lower denomination (except the lowest) from the number (less one) of that denomination necessary to produce the next higher one (thus, 15*s.* are subtracted from 19*s.* to produce 4*s.*), and the lowest denomination from the complete number of that denomination necessary to produce the next superior one.

The result is called the *compound complement* of the original compound number.

Ex. Compound Numbers.	Compound Complements.
53*l.* 2*s.* 9¼*d.*	46*l.* 17*s.* 2¾*d.*
6*l.* 16*s.* 2*d.*	3*l.* 3*s.* 10*d.*
2 cwt. 1 qr. 5 lbs. 3 oz.	7 cwt. 2 qr. 22 lbs. 13 oz.
526 miles 5 furlongs.	473 miles 3 furlongs.

71. The sum of a compound number and its complement will be a number of the highest denomination only, and will be 1 followed by as many zeroes as there are digits in the highest denomination.

Thus the sum of the pairs in the last examples are 100*l.*, 10*l.*, 10 cwt., 1000 miles respectively.

72. *Compound supercomplement.* This is formed by increasing each *digit* of the highest denomination, and each *number* of the lower denominations by 1, carrying where necessary.

Ex. The supercomplements of the numbers in the first column of ex. in (**70**) are

57*l.* 18*s.* 4*d.*	4*l.* 4*s.* 11*d.*
8 cwt. 3 qr. 23 lbs. 14 oz.	584 miles 4 furlongs.

73. The sum of a number and its supercomplement consists of a series of *ones*—namely, one more ones than there are digits in the highest denomination, and one of each lower denomination.

Thus the sums of the numbers in (**70**) and their supercomplements in (**72**) are respectively

111*l.* 1*s.* 1¼*d.*	11*l.* 1*s.* 1*d.*
11 cwt. 1 qr. 1 lb. 1 oz.	1111 miles 1 furlong.

74. *Compound nonundecimal.* This is a number which, *when reduced to its lowest denomination,* is nonundecimal. It is therefore detected by reducing and finding the last sum and difference of the result. This last sum should be 9, and last difference 0.

75. To form a compound nonundecimal, take any 4-digit nonundecimal as a number of the lowest denomination, reduce it, and then add to the result any nonundecimal number of units of the highest denomination.

Thus take 3465 farthings. We have

4)3465

12)866.1

2,0)7,2.2

3*l.* 12*s.* 2¼*d.*

to which adding 3465*l.* or 2178*l.*, &c., we get 3468*l.* 12*s.* 2¼*d.* or 2181*l.* 12*s.* 2¼*d.*, &c., as compound nonundecimals.

76. Or we may take 99 of the lowest denomination, and reducing it once for all, take any multiple of the result for the lower denominations, and add any nonundecimal of the highest denomination to the result.

Thus 99 farthings are 2*s.* 0¾*d.* Hence multiplying by 2, 3, 4, 5, we get 4*s.* 1½*d.*, 6*s.* 2¼*d.*, 8*s.* 3*d.*, 10*s.* 3¾*d.* for the lower denominations, and hence 3465*l.* 4*s.* 1½*d.*, 2178*l.* 6*s.* 2¼*d.*, &c., are nonundecimal.

77. In the annexed table this calculation is performed for the principal lowest denominations, the multiplication proceeding as far as 10. The lowest complete number of the highest denomination containing a nonundecimal number of the units of the lowest denomination is also given. Any multiples, sum, or differences of any of these lines may then be taken and added to produce a new nonundecimal. To a nonundecimal thus formed we may add a nonundecimal number of the highest denomination, and thus obtain an infinite number of compound nonundecimals almost at sight. In the last line of each table are inserted, in smaller figures, the number of each smaller denomination which makes up one of the higher, thus we see that 12*d.* makes 1*s.*, and 20*s.* 1*l.*, &c.

Ex. Taking the sum of lines 4, 7, and 320 in (**78**) we obtain 34*l.* 2*s.* 8½*d.* as a compound nonundecimal, being 331 times 99 farthings. Now 3564 is a simple nonundecimal. Considering this, therefore, as *l.*, the highest denomination, and adding to the former result we have 3598*l.* 2*s.* 8½*d.* as a new compound nonundecimal.

Again, adding lines 10 and 3 in (**81**) we get 5*lb.* 0*oz.* 7*dr.*, which is a nonundecimal, being 13 times 99 drams. And hence, adding 396*lb.*, which is also a nonundecimal, we have 401*lb.* 0*oz.* 7*dr.* avoirdupoise, as a new compound nonundecimal.

The principle on which the Tables are formed being thus understood, it is easy to form one for any particular case that may be required. The Tables embrace the nine in most common use.

COMPOUND NONUNDECIMALS.

78. Money.

	£.	s.	d.
1		2	0¾
2		4	1½
3		6	2¼
4		8	3
5		10	3¾
6		12	4½
7		14	5¼
8		16	6
9		18	6¾
10	1	0	7½
320	33	0	0
		20	12

79. Troy Weight.

	lbs.	oz.	dwt.	gr.
1			4	3
2			8	6
3			12	9
4			16	12
5		1	0	15
6		1	4	18
7		1	8	21
8		1	13	0
9		1	17	3
10		1	21	6
640	11	0	0	0
		12	20	24

80. Apothecaries' Weight.

	lbs.	oz.	dr.	scr.	gr.
1			1	1	19
2			3	0	18
3			4	2	17
4			6	1	16
5		1	0	0	15
6		1	1	2	14
7		1	3	1	13
8		1	5	0	12
9		1	6	2	11
10		2	0	1	10
640	11	0	0	0	0
		12	8	3	20

81.—Avoirdupois Weight.

	lbs.	oz.	dr.		tons.	cwts.	qrs.	lbs.
1		6	3	1			3	15
2		12	6	2		1	3	2
3	1	2	9	3		2	2	17
4	1	8	12	4		3	2	4
5	1	14	15	5		4	1	19
6	2	5	2	6		5	1	6
7	2	11	5	7		6	0	21
8	3	1	8	8		7	0	8
9	3	7	11	9		7	3	23
10	3	13	14	10		8	3	10
256	99	0	0	2240	99	0	0	0
		16	16			20	4	28

82.—Long Measure.

	ft.	in.	eighths.		miles.	furl.	poles.
1	1	0	3	1		2	19
2	2	0	6	2		4	38
3	3	1	1	3		7	17
4	4	1	4	4	1	1	36
5	5	1	7	5	1	4	15
6	6	2	2	6	1	6	34
7	7	2	5	7	2	1	13
8	8	3	0	8	2	3	32
9	9	3	3	9	2	6	11
10	10	3	6	10	3	0	30
32	33	0	0	320	99	0	0
		12	8			8	40

83.—Land Measure.

	Acres.	roods.	perches.
1		2	19
2	1	0	38
3	1	3	17
4	2	1	36
5	3	0	15
6	3	2	34
7	4	1	13
8	4	3	32
9	5	2	11
10	6	0	30
160	99	0	0
		4	40

84.—Liquid Measure.

	Gals.	quarts.	pints.
1	12	1	1
2	24	3	0
3	37	0	1
4	49	2	0
5	61	3	1
6	74	1	0
7	86	2	1
8	99	0	0
9	111	1	1
10	123	3	0
8	99	0	0
	4	2	

Examples for School or Family Use.

COMPOUND ADDITION.

85. Write on the black board a series of compound numbers having the same number of digits in the highest denomination, with the complements of all but the first. Let the pupil write down any last number and add.

The sum will be the sum of the two numbers that are not complementary increased by adding on 1 before the last place on the left for each complementary pair. Consult (45).

	£	s.	d.	
Ex.	562	12	9¾	any number.
	683	7	2½	numbers.
	264	11	10	numbers.
	593	9	9½	numbers.
	316	12	9½	complements.
	735	8	2	complements.
	406	10	2¾	complements.
	256	3	2½	number supplied by pupil.
Sum	3818	16	0	

	£	s.	d.	
which is the sum of	562	12	9¾	the first line
and	256	3	2¼	the last line
or	818	16	0	

increased before the last left-hand place by 3, the number of complements.

86. Write down a series of compound numbers, having the same number of digits in their highest denomination, with the complements of all but the last, and the supercomplement of that. Write any last line, and add.

The sum is found by increasing each digit of the highest denomination, and each number of the lower denominations in the last line by 1, and prefixing the number of complements, including the supercomplement. Consult (46), the "proof number" being in this case a series of ones.

	£	s.	d.	
Ex.	683	7	2¼	*a*
	264	11	10	*b*
	593	9	9¼	*c*
	316	12	9¼	complement of *a*
	735	8	2	,, of *b*
	517	11	4	supercomplement of *c*
	256	3	2¼	
	3367	4	3½	
which is	256	3	2¼	or last line
added to	3111	1	1¼	

87. When it is suspected that the pupil is inclined to *force* a result by knowing the mode of forming the sum, or by adding the constant numbers once for all, the teacher must have recourse to compound nonundecimals. In this case the result must be reduced to the lowest terms used in any of the lines added, and this reduced number will be nonundecimal. Such examples therefore also furnish convenient examples for *Reduction*.

Ex. From the table in (78) we may form the compound nonundecimals 6 and 320.

£	s.	d.	
33	12	4 ½	sum of lines 6 and 320
67	0	7 ½	sum of line 10 and twice line 320
1	15	0 ¾	sum of lines 10 and 7
3466	0	7 ½	sum of 3465*l.* and line 10
3568	8	8 ¼	
20			
71368			
12			
856424			
4			

3425697 which is nonundecimal.

In this case the nonundecimals are placed on the board, and some are selected and added. The result may be added to others, and so on, taking care to make the total a part of each new addition. The proof by reduction being applied to the last total will prove the correctness of all the totals. This is, therefore, a convenient form for self-practice.

Compound Subtraction.

88. Write on the black board a series of compound numbers (all having the same number of digits in the highest denomination), and the complements of all but the last. Let the pupil select any compound number containing the same number of digits in the highest denomination, preceded by a digit representing the number of numbers. From this as a minuend subtract the series of numbers on the board.

The sum of the last difference and the *middle* subtrahend will reproduce the original minuend, with its first left hand digit diminished by the number of complements. Consult (55).

Ex. Write on the board—			(This column is not to be written.)
£	*s.*	*d.*	
6 8 3	7	2 ½	*a*
2 6 4	11	10	*b*
5 9 3	9	9 ¼	*c* (middle number)
3 1 6	12	9 ½	complement of *a*
7 3 5	8	2	complement of *b*

Here there are *three* numbers, *a*, *b*, and *c*, hence 3 is to be the prefixed digit. Suppose the minuend selected to be 3 6 3 2*l.* 9*s.* 5¾*d.* Then the process becomes

	£	*s.*	*d.*	
From	3 6 3 2	9	5 ¾	original minuend
take	6 8 3	7	2 ½	
from	2 9 4 9	2	3 ¼	
take	2 6 4	11	10	
from	2 6 8 4	10	5 ¼	
take	5 9 3	9	9 ¼	middle subtrahend
from	2 0 9 1	0	8	
take	3 1 6	12	9 ½	
from	1 7 7 4	7	10 ½	
take	7 3 5	8	2	
	1 0 3 8	19	8 ½	last remainder

The sum of this last remainder and middle subtrahend is 1 6 3 2*l.* 9*s.* 5¾*d.*, which is the original minuend with its first left hand digit (3) diminished by the number of complements (2).

For a new example, change the original minuend. At commencing, only select 2 numbers and 1 complement, instead of 3 numbers and 2 complements.

Compound Multiplication.

89. Write down on the board a series of numbers and their complements. Multiply any number and its complement by the same multiplier, the sum of the products will be the multiplier followed by zeroes. Consult (5 6).

Ex. 1.

£	s.	d.		£	s.	d.
683	7	2½		316	12	9½
		7				7
4783	10	5½		2216	9	6½

the sum of which products is £7000, or the multiplier 7 followed by zeroes.

Ex. 2. To multiply by 254.

	£	s.	d.			£	s.	d.	
a...	3	6	2½	1 *a*	*a*...	6	13	9½	1 *a*
	33	2	1	10 *a*		66	17	11	10 *a*
	331	0	10	100 *a*		668	19	2	100 *a*
	13	4	10	4 *a*		26	15	2	4 *a*
	165	10	5	50 *a*		334	9	7	50 *a*
	662	1	8	200 *a*		1337	18	4	200 *a*
	840	16	11			1699	3	1	

The multiplicands being compound complementaries, the sum of the products is £2540—that is, the multiplier 254 followed by a zero.

Compound Division.

90. By subtracting 1 unit of the lowest denomination from a compound complement, we form a compound *sub*-complement. On prefixing to a pair of such subcomplementary compound numbers two simple numbers, whose sum is one less than the divisor, and then using the results as dividends, the quotients will be subcomplementary compound numbers, and the sum of the remainders will be one less than the divisor.

Hence, prefixing these remainders to the quotients, we obtain new examples. Consult (61).

Ex. 683*l.* 7*s.* 2½*d.* and 316*l.* 12*s.* 9½*d.*, being complementary, we have only to write ¼*d.* for ½*d.* in the last to make them subcomplementary. If we select 7 as the divisor, we must prefix two numbers, as 2 and 4, whose sum is 6, or one less than the divisor 7. Thus—

	£	s.	d.			£	s.	d.	
7)	4683	7	2½	and	7)	2316	12	9¼	
a...	669	1	0¼	...3	*b*...	330	18	11¼	...3

in which the quotients are subcomplementary, producing when added 999*l.* 19*s.* 11¾*d.*, and the sum of the remainders, 3 and 3, is 6, or 1 less than the divisor. Prefixing these to the quotients, we have the new examples—

	£	s.	d.			£	s.	d.	
7)	3669	1	0¼	from *a*	7)	3330	18	11¼	from *b*
c...	524	3	0¼	...1	*d*...	475	16	11¾	...5
whence 7)	1524	3	0	from *c*	7)	5475	16	11¾	from *d*
	217	14	8½	...2		782	5	3¼	...4

and so on for ever.

91. The same process would answer for long division, but the use of nonundecimals is here much simpler, as each separate example admits of immediate verification.

Select a compound nonundecimal from the table. Select a simple nonundecimal as the divisor. The remainder will be nonundecimal.

Ex. 1.

£	s.	d.	
	19	0	0½ quotient
396) 7525	0	7½	dividend
396			
3565			The dividend is formed by adding 7524*l.*, a compound nonundecimal, to 1*l.* 0*s.* 7½*d.*, given in Table (78), line 10.
3564			
1			
20			
20			
12			
247			
4			
990			
792			

Remainder 198 which is nonundecimal.

Ex. 2.

	lbs.	oz.	dr.	avoirdupoise
	6	5	7	quotient
297)	1883	11	5	dividend
	1782			
	101			
	16			
	617			
	101			
	1627			
	1485			
	142			
	16			
	857			
	142			
	2277			
	2079			

The dividend was formed by adding 1881 lbs. (a nonundecimal) to 2lbs. 11 oz. 5dr. given in the table (81), line 7.

Remainder 198 which is nonundecimal as before.

Ex. 3.

	acres.	r.	p.	
	5850	2	3	quotient
2673)	15638441	2	11	dividend
	13365			
	22734			
	21384			
	13504			
	13365			
	1391			
	4			
	5566			
	5346			
	220			
	40			
	8811			
	8019			

The dividend is formed by adding 15638436 a. (a nonundecimal) to 5a. 2r. 11p. given in the table (83), line 9.

Remainder 792 which is nonundecimal.

SELF-PRACTICE FOR ADVANCED COMPUTERS.

1.—*Contracted Division.*

92. *Guy's Rule.* (Pronounce *Guy* as *gee* in *geese*.) Suppose that the dividend consists of a significant number, followed by a succession of zeroes, and that the division has been performed in the ordinary manner until all the significant digits in the dividend have been exhausted, and none but zeroes remain to be brought down. Then several additional figures in the quotient may be obtained *with the certainty that all but the last are correct, and that the error in the last figure of the quotient does not exceed one*, by the following contracted method (*App.* **35**).

Instead of bringing down a zero, cancel the last figure of the divisor, and use the remainder as a new divisor. Multiply this divisor by the new digit in the quotient, taking care to multiply the cancelled digit, and carry its *tens only* (paying no regard whatever to the units) to the units of the first uncancelled digit of the divisor, and subtract this product for a new partial dividend.

Cancel another figure to the right of the divisor, and proceed as before.

Continue the process until, by successive cancellings, the divisor is so reduced that it only just exceeds *twice the number of digits cancelled*. This must be the last mutilated divisor. If you proceed further, the divisor will be less than twice the number of digits cancelled, and the limit of error cannot be foreseen. See the *Example* in (**93**.)

The important advantage of this rule over the ordinary one is, that it gives the limit of error with certainty.

93. Take any two numbers, of considerable size, and add them. Use their sum as a divisor, and each of them with an indefinite number of zeroes annexed as dividends. Determine as many digits in the quotient as possible, by the rule in (**92**). Then *up to the last right hand digits* at least, the quotients will be subcomplementary, and they may be entirely so. (*App.* **36**.)

Ex. Take 12365485400000000 } as dividends
36829344500000000 }

and the sum 491948299 of their significant digits, as divisor.

```
                                    2513574
491948299)1236548540000000
          98389659 a
          ---------
          25265195
          24597414 b
          ---------
            667781
            491948 c
            ------
            175833
            147584 d
            ------
             28249
             24597 e
             -----
              3652
              3443 f
              ----
               209
               196 g
               ---
                13
```

```
                                    7486425
491948299)3682934450000000
          3443638090 a'
          ----------
           239296360
           196779810 b'
           ---------
            42517050
            39355850 c'
            --------
             3161200
             2951680 d'
             -------
              209520
              196770 e'
              ------
               12750
                9830 f'
                ----
                2920
                2450 g'
                ----
                  47
```

Here the quotients are subcomplementary. The practised computer will of course perform the multiplication and subtraction in one operation (62.) The lines *a*, *b*, *c*, *d*, &c., have been merely written in to show the operation.

As the divisor is already larger than the dividend, a 0 would have had to be brought down at first; instead of doing so, therefore, we cancel (or, in printing, overdot) the last digit of the divisor, and place the digit of the quotient over the number which *would* have otherwise been brought down. Line *a* is formed by multiplying 4 9 1 9 4 8 2 9, the mutilated divisor, by 2, adding in 1, for the *tens* carried from 18, the product of the *cancelled* 9 by 2. Having obtained the difference, we cancel the next 9 in the divisor, reducing it to 4 9 1 9 4 8 2, and multiply it by 5 to form *b*, carrying 4, being the tens of 45, the product of the cancelled 9 by 5. We proceed in this way till we reach 4 9 as a divisor. We have now cancelled 7 digits of the divisor, and as 4 9 is greater than twice 7 or 14, while if we cancelled the 9, and reduced the divisor to 4, this 4 would be *less* than twice 8 (the cancelled figures), the rule determines 4 9 to be the last mutilated divisor that can be taken. Hence only 7 digits could be determined by the contracted method, and if more were required, we must determine the others first by complete division, and then use the contracted method.

94. Other examples may be formed by taking any small number, as 4 3, and multiplying it successively by 1, 2, 3, 4, 5, &c. Assuming the results, with an indefinite number of zeroes subjoined, as dividends, and dividing them all by any large number as a divisor. The quotients divided respectively by 1, 2, 3, 4, 5, &c., will all give the same number (*App.* **37**).

Thus if 4 3, 8 6, 1 2 9, 1 7 2, 2 1 5, &c., followed by zeroes, be taken as dividends, and 1 5 6 7 4 as the common divisor, which will allow of 3 digits in the quotients being determined in the contracted way, the quotients are 2 7 4, 5 4 8, 8 2 3, 1 0 9 7 (the first digit is here found by complete division), 1 3 7 1 (the first digit by complete division), &c., and if these are divided by 1, 2, 3, 4, 5, &c., the result is 2 7 4 in all cases. Observe that the quotients are not precise multiples of 2 7 4, and hence a proof by multiplication would not succeed.

Or, taking any considerable number, as 1 5 6 7 4, multiply it in succession by 1, 2, 3, 4, 5, &c., and taking the results as divisors, divide any common dividend by them; then the first quotient divided by 1, 2, 3, 4, 5, &c., will give all the quotients (*App.* **37**).

Thus multiplying 1 5 6 7 4 by 1, 2, 3, 4, &c., we obtain 1 5 6 7 4, 3 1 3 4 8, 4 7 0 2 2, 6 2 6 9 6, &c., as divisors; and if 4 3 followed

by any number of zeroes be the common dividend, and we determine 4 places of the first and second quotients (the first digit in each case by complete division) we obtain the quotients 2743, 1371, 914, 685, &c., which are all obtained by dividing 2743, by 1, 2, 3, 4, &c.

Similarly constructed examples are applicable for complete division.

II.—*Square Root.*

95. Table of the Squares of Simple Digits :—

Number	1,	2,	3,	4,	5,	6,	7,	8,	9
Square	1,	4,	9,	16,	25,	36,	49,	64,	81

96. Take any small number and multiply it by 4, add on to the number and to this product any even number of zeroes, and extract the square root of both resulting numbers. The larger root divided by 2 gives the smaller root, and if the larger root be even, the corresponding remainder is four times the other remainder (*App.* **38**).

Ex. Take 3, and 4 times 3, or 12, and adding on 6 zeroes to each, extract the square root.

	1	7	3	2	root						3	4	6	4	root						
	3,	0	0,	0	0,	0	0	radicand			12,	0	0,	0	0,	0	0	radicand			
	1										9										
27)	200									64)	300										
	189										256										
343)	1100									686)	4400										
	1029										4116										
3462)	7100									6924)	28400										
	6924										27696										
	176				remainder						704				remainder						

Here the second root is twice, and the second remainder four times the first respectively. The same is true for all the remainders except the first.

97. When the additional figures of the radicand are all zeroes, we can obtain additional figures of the root by *contracted division*, using

the last partial divisor, after doubling its last digit, as the divisor, and cancelling a digit in lieu of bringing down a period of two zeroes (*App.* **39**). There is the same restriction as to the number of digits in the root that can be determined in this way, as there is for the number of digits in the quotient that can be determined by contracted division (**92**). When the examples are chosen, as in (**96**), the verification will not extend to the *remainders* of the contracted division.

Ex. 1. Continuing the Examples in (**96**), after preparing the partial divisors, we have

```
  . .      05 root          . . .     102 root
3464)176              6928)704
      173                    692
     ---                    ---
       3                     12
```

in the last case, as 6 is just equal to twice 3, the number of digits cancelled, we are able to take it as the last divisor. Hence we find 173205 as the root of 30000000000 and 3464101 as that of 12000000000000.

Ex. 2. Take 15, and 4 times 15, or 60.

```
  3  8  7  2  9|833 root      7  7  4  5  9|6669 root
 15,00,00,00,00               60,00,00,00,00
  9                           49
 ---                         ---
68)600                      147)1100
   544                         1029
  ----                        -----
767)5600                    1544) 7100
    5369                        6176
   -----                       -----
7742)23100                  15485) 92400
     15484                      77425
    ------                     ------
77449)761600                154909)1497500
      697041                     1394181
  . . ------                 . . ------
77458) 64659                154918) 103319
         2693                       10369
          270                        1075
           28                         146
                                        8
```

And one half of 77459666 is 38729833.

The bar in the root marks the commencement of the contracted division, in which only the remainders are written (**63**).

III.—*Cube Root.*

98. Table of the Cubes of Simple Digits—

Numbers	1,	2,	3,	4,	5,	6,	7,	8,	9
Cubes	1,	8,	27,	64,	125,	216,	343,	512,	729

99. *Horner's Rule.* This, the only practical, and almost only practicable method of extracting the cube root, is best explained by an example (*App.* **40**).

Ex. Extract the cube root of 498865935491.

| | | | | | | | |

First write down 1, 0, 0 0, and the radicand Z at the head of four columns.

Divide the radicand into periods of three, beginning on the right and counting to the left.

Find from the table (9 8) the nearest cube (7), or A, to the first period.

Form C by multiplying 1, the last number in col. V by A (giving B), and adding the result to 0, the last number in col. X.

Form E by multiplying C by A (giving D), and adding the result to Y.

Form G, 155, by multiplying E by A (giving F), and *subtracting* the result from the first period.

Add on H (the next period) to G, giving (z).

Now begin again, and

Form L by multiplying V by A (giving K), and adding the result to C.

Form N by multiplying L by A (giving M), and adding the result to E.

Add on P (two zeroes) to N, giving (y).

Do nothing in the last column this time, but begin again, and

Form R by multiplying V by A (giving N), and adding the result to L.

Add on S (one zero) to R, giving (x).

Do nothing in the two last columns, but begin again, and

We have now completed the preparation for the next stage (by a process exactly equivalent to doubling the last digit in the rule for square root), and have four columns beginning with V, x, y, z, instead of V, X, Y, Z.

Determine (a) the next digit in the root by dividing (z) by (y), taking care not to take (a) too large. If (a) be taken too large, (f) will be greater than (z), and if too small, (g) will be greater than (y).

Form (b, c, d, e, f, g, h, k, l, m, n, p, q, r, s, x, y, z) in precisely the same way as before, reading the preceding instructions with (b, c, d, ... x, y, z) in place of (B, C, D, ... x, y, z).

Determine a by dividing z by y, and proceed as before to form (b, c, d, ... x', y', z').

Determine a' by dividing z' by y', and proceed as before. But here we find g' to be zero, and hence we have completed the extraction of the root.

In the preceding example many unnecessary figures have been introduced in order to make each stage of the process perfectly uniform and intelligible, the order of the letters of the alphabet

distinctly marking out the stages. But a skilful computer, who can perform the multiplication of a number by a single digit, and its addition to or subtraction from another number in a single operation (32, 34, 36), need not write down more than the following figures where the letters (which are of course omitted in practice) point out the corresponding numbers in the complete work.

				A	a	a	á
				7	9	3	1
C	7	E	49	Z	493,865,935,491		
H	14	y	14700	z	155 865		
x	210	e	16671	z	5 826 935		
c	219	y	1872300	z′	188 678 491		
h	228	e	1879419	g′			0
x	2370	y′	188654700				
c	2373	e′	188678491				
h	2376						
x′	23790						
c′	23791						

Even the practised computer should always make adding on the zeroes and bringing down the periods, a separate operation to mark the completion of a stage and the conclusion of the operations in any column. Less practised computers can condense the first column, but had better work out the others at length, as in p. 54. See this "Semi-condensed" form employed in *Ex.* 1, p. 59.

100. Take the first 12 digits of any number in col. A of the *Auxiliary Table*, and extract the cube root. The roots are given in (*App.* **21.**)

101. Take any small number and multiply by 8. To the number and the product add on zeroes in any number divisible by 3. Extract the cube roots of each result. The larger root divided by two gives the smaller root, and if the larger root is even, the corresponding remainder is eight times the other remainder *App.* (38).

Ex. Since 8 times 562 is 4496.

		8	2	5			1	6	5	0
8	64	562,000,000			1	1	4,496,000,000			
16	19200	50000			2	300	3496			
240	19684	10632000			30	516	400000			
242	2017200	484375			36	76800	3875000			
244	2029525				42	79225				
2460					480	8167500				
2465					485					
					490					
					4950					

Only the condensed form of the operation is here given. The first root, 8 2 5, is half the second, 1 6 5 0, and the first remainder, 4 8 4 3 7 5, is one eighth the second, 3 8 7 5 0 0 0.

102. When the periods to be brought down consist only of zeroes, many additional figures of the root may be obtained by a contracted process similar to, and finally ending in, a contracted division. The exact process will be easily obtained from observing the following examples (*App.* **39**).

Continuing the first example in the last article, which requires preparation for the next stage, we complete the third stage.

The last numbers were

23715 root

	2465		2029525	*z*	484375
	2470	*a*	2041875		
b	|24|75	*c*	204236	*d*	75903
		e	20$\dot{4}\dot{2}\dot{8}$|5	*f*	14619
				h	820
				i	116
				k	16

The completion of the third stage by the uncontracted process, introduces the numbers *b* and *a*. Then, instead of adding on one zero to the first and two zeroes to the second column, and bringing down three digits in the last column, we bar off two digits in the first and one in the second column, and bring down none in the third, which is tantamount to adding on the zeroes as usual, and then barring three digits in each. Determine 2, the next root digit, by dividing *z* by 2 0 4 1 8 7, the remnant of *a*. Multiply 2 4, the remnant of *b*, by this digit 2 (carrying 1 from the tens arising from multiplying the cancelled 7 by 2), and add to *a* to produce *c*. Multiply *c* by 2 and subtract the result from *z* to produce *d*. To complete this fourth stage we multiply 2 4, the remnant of *b*, by 2, (carrying 1 as before,) and add to *c* to produce *e*.

Next bar off one digit in *e* and two in *b*, which now disappears, and determine the next digit of the root by dividing *d* by this remnant of *e*. As there would be no tens produced by multiplying 2, the last cancelled figure in *b*, by 3, we have not to prepare *e* any further, but simply to multiply its remnant, 2 0 4 2 8, by 3, and subtract from *d* to give *f*.

The other 3 digits of the root, 7 1 5, are found by common contracted division, of which only the remainders, *h*, *i*, *k*, are written.

Continuing the *second* example in the last order, we observe that as the last root digit was 0, the last numbers need no further preparation to complete the third stage, and the process will be—

474306 root.

b	4|9|50	*a*	816750|0	*z*	3875000
		c	816948	*d*	607208
		e	81714|6	*f*	85192
		g	81̇7̇1̇6̇	*h*	2506
				i	55
				k	7

Where the letters refer to the former explanation, the only difference being that as 7 times 4 (the first figure of the cancelled pair 49 in *b*) gives two tens, we have to add this to the remnant of *e*, producing *g* before commencing the contracted division.

We thus find that the cube roots of 562 and 4496, with the proper number of zeroes added on, are 82523715 and 1650474306, the first being half of the second so far as the digits extend.

103. The operation of extracting the cube root by Horner's rule, writing down the work in the condensed form of (101), and obtaining extra digits by the contracted division as in (103,) is such an excellent exercise in all the processes of computation, that those who aspire to be ready computers should work at least a pair of the examples in (101) daily; obtaining from 10 to 20 places of figures in the root. It is for this reason principally that they have been introduced into this manual of examples.

To render the process still more intelligible, four examples are added. The first is worked out in the semi-condensed form mentioned on p. 56 as suited for less practised computers, and the remainder in the condensed form recommended for daily practice. To avoid writing a number of zeroes, the roots are treated as decimals, with which those who have proceeded thus far are probably acquainted, but if not, they have only to suppose three times as many zeroes added on to the radicand as there are digits after the point in the root, and then omit the point. The *bar* in the root marks the commencement of the contraction, and the star shows where the contracted division begins.

The learner should work each example independently by the semi-condensed method, and then verify every sum and difference by the condensed example, and account for every figure on the principles here explained. He should then re-work them in the condensed method. The two latter examples, where upwards of 20 figures of the root are determined, show the immense superiority of *Horner's Rule* over all others. An example of its application to the extraction of the *fifth* root is given on p. 72.

Ex. 1. Semi-condensed form. Cube root 4·7026|6937 5441

C	4	E	16	of 104	*
H	8	M	32	F	64
X	120				
c	127	y	4800	z	40000
h	134	d	889	f	39823
x	14100				
	14102	e	5689	*z*	177000000
	14104	m	938		132596408
	141060	y	66270000	*b*	44403592000
	141066		28204		39800925576
	141072				
	14|10|78		66298204	*c*	4602666424
			28208		3980651202
			66326412 00		622015222
			846396		597106440
			66334875 96		24908782
			846432		*19903587
			66343340 2|8		5005195
			8464|2		4644170
			66344186 7		361025
			8464		331726
			66345033|1		29299
			126|9		26538
			66345160		2761
			126|7		2653
			6634528|67		108
			42		66
			66345290|9		42

The letters in this and the following example refer to the operations explained in pp. 54, 55.

Ex. 2. Condensed form. Cube root 2·3513|34687721

C	2	E	4	of	13 *
H	4	y	1200	z	5000
x	60	e	1380	z	833000
c	63	*y*	158700	*a*	22125000
h	66		162175	*b*	5550449000
x	690		16567500	*c*	575333303
	695		165745 51		77751905
	700		165816030 0		11406765
	7050		165837189 9		*1454979
	7051		165858350|7		128075
	7052		165860466		11971
	70530		165862572		361
	70533		16586285		30
	70536		165863 31|3		14
	7|05|89				

The root in *Ex.* 2 is exactly half that in *Ex.* 1, and the remainders *z a b c* in *Ex.* 1, are 8 times the remainders *z s b c* in *Ex.* 2 respectively.

Ex. 3.	Cube root 4·30886938|006376744351842	
4	16 of 80	*
8	4800 a 16000	
120	5169 m 493000000	
123	55470000 b 48413888000	
126	55573264 c 3864342528000	
12900	5567659200 d 522459653544000	
12908	5568693184 n 21169194901091000	
12916	556972723200 p 4459476984451443000	
129240	556980479076 e 3551787045354328	
129248	55698823498800 f 209843085205510	
129256	5569893938101 42745887195627	
1292640	556990561774830 0 3756540993289	
1292646	5569906005546519 g^* 4145970330 89	
1292652	55699063933447470 0 24703571066	
12926580	5569906496757610 84 h 2423944665	
12926589	5569906600170475|3|2 1959320 25	
12926598	556990660024803|1 28885827	
129266070	55699066003255|8|7 1036297	
129266073	5569906600329|4|5 479307	
129266076	556990660033|3|3 23715	
1292660790	55699066003|3|4 1436	
1292660798	323	
1292660806		
1|2|9|2|6 6|0 8|1 4		

Ex. 4.	Cube root 2·15443469|003188372175 92	
2	4 of 10	*
4	1200 a' 2000	
60	1261 m' 739000	
61	132300 61625000	
62	135475 b' 6051736000	
630	13867500 c' 483042816000	
635	13893316 d' 65307456693000	
640	1391914800 n' 96085297814960 00	
6450	1392173296 p' 12536729060662640 00	
6454	139243180800 e' 4483973380669291	
6458	139245119769 f' 26230385650689	
64620	13924705874700 1230561914986 9	
64624	13924731727876 g' 1165805949209	
64628	1392475758106800 51824629144	
646320	139247614590 4956 10050329644	
646323	1392476533703 14800 h' 3029930 94	
646326	1392476591872 88301 24497644	
6463290	139247665004261|8|8|3 10572878	
6463294	13924766500620 0|7 825542	
6463298	1392476650081 3|9 129304	
64633020	1392476650082 0|3 3985	
64633026	139247665008 26|7 1301	
64633032		
646330380		
646330389		
646330398		
6|4 6|3 3|0 4|0 7		

The cube root in this case is one half that in *Ex.* 3. The remainders a, b, c, d, e in *Ex.* 3 are 8 times the remainders a', b', c', d', e' in *Ex.* 4 respectively, as in all these cases the corresponding root in *Ex.* 3 is exactly double that in *Ex.* 4. After the contraction commences, this relation of the remainders does not hold for their latter digits. Compare remainders f, g, h in *Ex.* 3 with the remainders f', g', h' in *Ex.* 4. These relations form convenient stages of verification in so long an operation. The remainders m', n', p' do not correspond to m, n, p, for the reason mentioned in (*App.* **38**).

APPENDIX.

FOR THE USE OF TEACHERS AND PARENTS WHO ARE ACQUAINTED WITH ELEMENTARY ALGEBRA.

1. The subcomplement of $a=(10^n-1)-a$, where n is the number of digits in a.

2. If N be a binonal whose base is a, and extent n, then by this definition

$$N=a.\ 10^n+(10^n-1)-a=(a+1).\ (10^n-1).$$

Hence $b.\ (10^n-1)$ is a binonal of the base $b-1$ and extent n, provided n be not less than the number of digits in $b-1$.

3. Let $N_1\ N_2\ldots N_m$ be binonals of the same extent n, and of the bases $a_1,\ a_2\ldots a_m$.

Then as $N=(a+1)\ (10^n-1)$ by (*App.* **2**).

$$N_1+N_2+\ldots+N_m=(a_1+a_2+\ldots+a_m+m)\ (10^n-1)$$

which will be a binonal of the same extent, provided n be not less than the number of digits in $a_1+a_2+\ldots+a_m+m-1$, by (*App.* **2**). But if m do not exceed 10, the number of digits in the sum $N_1+N_2+\ldots+N_m$ cannot exceed $2n$, provided none of the numbers $N_1,\ N_2\ldots N_m$ have more than $2n-1$ digits; that is, provided there be at least one more digit in the second part of each binonal than the first, which will be caused by the interposition of at least *one* nine between each number and its subcomplement.

Mr. T. Tate, who first gave the binonal system of constructing examples in his excellent little work on *Arithmetic*, (pp. 100-102), but without the nomenclature here adopted, has accidentally misstated this limitation. He says,—

"*Ex.* 1 3, 8 6 — *Addition.* Here in each row the corresponding figures on the left hand added
3 4, 6 5 — to those on the right produce nines; and
2 0, 7 9 — the same law is observed in the answer.
5, 9 4 — The only precaution is simply that the sum
7 5, 2 4 — of the last column should not exceed 9."

Now if we add	Where the sum of the last	
49, 50	column is 9, the result does	49, 950
56, 43	not follow the law. The pre-	56, 943
105, 93	caution of inserting a 9 is, however, effectual, thus—	106, 893

4. Let $N=(a+1)(10^n-1)$ as before;

then $MN=M(a+1)(10^n-1)$,

which will be binonal and of the base $\left\{M(a+1)\right\}-1$, provided this number have no more than n digits (*App.* **2**). Let M have m digits, and therefore be less than 10^m; then as $10^m(a+1)$ has m more digits than $a+1$, $M(a+1)$ will certainly have no more than n digits, and MN will be binonal provided that $a+1$ or a have no more than $n-m$ digits, and therefore there be m more digits in the second part of N than in the first; that is, provided that N have m middle nines, or one for each digit in the multiplier.

Mr. Tate has here also not stated the limitations correctly. Thus, in giving as an example $256,9,743\times34=8737,1262$, he says,—"Here the figures to the left and right of the nine are formed in the same way as in addition, and the figures in the product follow the same law. Where there are 3 figures in the multiplier there must be 2 central nines, and so on. The only precaution is simply that the product of the 2 by the 3 does not produce a greater number than 9." Now both these conditions are fulfilled by $387,9,612\times39=15130468$; but the law does not hold for the product. On inserting *two* middle nines however, we have $387,99,612\times39=151331,84868$, in which the law does hold.

Mr. Tate refers to the "English Journal of Education" for 1845 for the demonstration of his rules, but as the present writer has not seen this demonstration, he is unable to point out the source of error.

5. Let N_1 and N_2 be binonals, as in (*App.* **3**),

then $N_1-N_2=(a_1-a_2)(10^n-1)$,

and as a_1 and a_2 have each no more than n digits, their difference can have no more, and hence the result is a binonal. (*App.* **2**.)

Generally if $M=a\,.\,10^n+c-a$

and $M'=a'\,.\,10^n+c-a'$ where c is any number

$M-M'=(a-a')(10^n-1)$

which will be binonal.

6. The complement of a number a, having n digits, is 10^n-a.

7. The supercomplement of a number a, having n digits, is $10^n - a + \frac{1}{9} \cdot (10^n - 1)$.

8. Let the digits of any number N be $a_0\ a_1\ a_2 \ldots a_n$

so that $N = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + \ldots + a_n \cdot 10^n$

$= a_1 \cdot (10-1) + a_2 \cdot (10^2-1) + \ldots + a_n \cdot (10^n-1)$

$+ a_0 + a_1 + \ldots + a_n$

and as $10-1$, 10^2-1, $\ldots$ 10^n-1 are each divisible by 9, the remainder, after dividing N by 9, will be the same as that after dividing $a_0 + a_1 + \ldots a_n$ by 9. But if this sum have more than one digit, the remainder, after dividing *it* by 9, will be the same as that after dividing the sum of *its* digits by 9. And so on, until we come to a sum of digits which is less than 10, and this, if 9, is divisible by 9 without remainder, and in that case the original number N is so divisible; but if *not* 9, must be the remainder after dividing itself by 9, and therefore the remainder after dividing N by 9.

9. This rule is an immediate deduction from observing the long division of a number by 11. As the rule given in the text is exactly identical with common subtraction, it saves the algebraic subtraction with negative numbers occurring in Prof. De Morgan's rule, and it is, therefore, better adapted to the use of beginners. Observe the process, the quotient being written over the digits of the dividend as they are brought down—

```
        7 7 8 4 8 0
11 ) 8 5 6 3 2 8 4
     7 7
     -----
       8 6
       7 7
       -----
         9 3
         8 8
         -----
           5 2
           4 4
           -----
             8 8
             8 8
             -----
               0 4
               0 0
               -----
                 4
```

The left hand place of each remainder here is necessarily the difference or increased difference of the process by the rule (**14**, *Ex.* 1,) and the rule consequently gives us the true remainder at last.

10. Let M and N be nonundecimals, and let $M=99m$ and $N=99n$, where m and n are some smaller numbers. Then

11. $M+N=99m+99n=99\ (m+n)$

12. $M-N=99m-99n=99\ (m-n)$

13. $MN=99\ .\ 99mn$

14. and if $\frac{M}{N}=\frac{99m}{99n}=a+\frac{b}{99n}$

where a is the quotient, and b the remainder, $b=99\ (m-a)$ or is also nonundecimal.

Hence the sum, difference, and product of nonundecimals is nonundecimal, as is also the remainder resulting from the division of any nonundecimal by any other nonundecimal.

15. As $Na=99na$ is also nonundecimal, the product of a nonundecimal by any number whatever is nonundecimal. Hence we may add on zeroes at pleasure after a nonundecimal, and the result will still be nonundecimal. This is a convenient way of increasing the dividend in long division, and as each new dividend formed by adding on a zero is nonundecimal, *all* the remainders from first bringing down a zero will be nonundecimal. (**64.** *Ex.*)

16. If N be a binonal with an even extent $2n$ and base a, $N=(a+1)\ (10^{2n}-1)=99\ (a+1)\ (10^{2n-2}+10^{2n-4}+\ldots+1)$ and is, therefore, nonundecimal. This property allows of our writing down nonundecimals as easily as binonals.

17. For $9999=1\ .\ (10^4-1)$, which is a binonal of base 0 and extent 4, being in fact 00009999. Again $99=1\ .\ (10^2-1)$ is a binonal of base 0 and extent 2, being in fact 0099. These are the two extremes, and the only two cases that might occasion doubt. In any other case the binonal is $(a+1)\ .\ (10^2-1)=99\ .\ (a+1)$ or a nonundecimal at once. Hence, if we make the divisor (in *App.* **15**) less than 9999, the remainders will be binonal.

18. Thus, if we divide one nonundecimal by another nonundecimal, the remainder should be nonundecimal (*App.* **14**). If we find it to be so, we may consider this as sufficient proof of the correctness of the work. But it is only a strong presumption, for the remainder might be multiplied by any number (*App.* **15**), or have any nonundecimal added to it (*App.* **11**), or subtracted from it, (*App.* **12**), and it would yet remain nonundecimal and therefore satisfy the conditions. But such changes may be considered as impossible to occur as errors of calculation. Hence we have a presumption approaching to certainty that the work is correct.

19. If $M=ma+b$ and $N=mc+d$, then

$$MN=m\left\{(ma+b)c+ad\right\}+bd$$

Hence the remainder on dividing MN by m will be the same as that on dividing bd by m. If m be 9, this gives the rule for last sums; if m be 11, for last differences. Again

$$\frac{M}{N}=\frac{ma+b}{mc+d}=mf+g+\frac{mh+k}{mc+d}$$

where $mf+g$ is the quotient, and $mh+k$ the remainder. Then

$$ma+b=m\left\{(mf+g)c+df+h\right\}+dg+k$$

Hence the remainder on dividing $dg+k$ by m will be b. If m be 9, this gives the rule for last sums, and if m be 11, for last differences.

The moral certainty resulting from the conditions being fulfilled for both last sums and differences arises from the considerations in *App.* **18**. If only one test is used, it should be that of last differences, as two successive digits must be *similarly* altered for the error not to be detected by last differences; whereas, any alteration of 9 in the sum of the digits, which would allow of their being arranged in almost any manner, would be left undetected by last sums.

20. The proofs of these exercises depend on the fact that there are 10 lines in each column in both tables, that the sum of any column in Table **1** is 10 times the first line, and that the sum of any column in Table **2** is 50 or 10 times the line containing 5. Table **2** may be increased indefinitely by making new columns of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 5, placed in any order.

For (**23**). The result of adding $10m$ to n is $10m+n$, where n is the number with which we commence, and m is the number of the column.

For (**24**). If n be the number added, the result for the line containing 5 is $5+n$, and the whole result is the addition of 10 times n to the sum of a column, or 50; that is, $50+10n=10\,.\,(5+n)$ or 10 times the result $5+n$ from the line containing 5.

In (**25**) we only add two columns or two 50's together, producing 100.

In (**25** *b*) we add together two columns each $=50$, less a single digit a; the result is, therefore, $100-a$.

In (**25** *c*) we add 10 times one column less 50, to another column less one digit (a), and each column being $=50$, the result is $(50.10-50)+(50-a)=500-a$.

In (26) we have only the inverse of (23).

For (27). If m be the constant subtrahend, and R the sum of the differences, and B the sum of the borrowed 10's, then, since the sum of any column is 50, and the sum of the subtrahends added to that of the remainders is the sum of the minuends, we have $10m+R=B+50$. Now as each column contains each digit once, except 5, which it contains twice, it follows that if m do not exceed 5, $B=10(m-1)$, and therefore $R=40$; but if m exceed 5, $B=10m$, and therefore $R=50$.

In (28) the sum of the numbers subtracted is 50, hence writing 50 for $10m$ in the last equation, it becomes $50+R=B+50$, or $R=B$.

In (29) the series of partial products is necessarily the same as the product of the whole sum by the common multiplier m; that is, $50m$, or $10.5m$.

In (30) if n be the constant addend, the sum of the results is clearly $50m+10n=10.(5m+n)$, while $5m+n$ is the result derived from the line containing 5.

In (31) the sum of the numbers added is 50; hence if the multiplier be m, we have for the result $50m+50=10.(5m+5)$.

In (32) the sum of the numbers added is clearly $50m+50+50=10.(5m+10)$.

In (33) the sum of all the numbers divided is $50.10+50=550$; hence the sum of the partial quotients and remainders must form the total quotient and remainder.

In (34), if R be the sum of the remainders, and B that of the borrowed tens, then, since the sum of any column $=50$, we have clearly, sum of subtrahends + sum of remainders = sum of minuends, or $(500+50)+R=B+50$, or $500+R=B$.

In (35), if m be the constant multiplier, and R and B as before, we have $(50m+50)+R=B+50$, or $50m+R=B$.

21. The construction of the *Auxiliary Table* is as follows:—

The first column gives the number of the line.

Column A contains a series of numbers, and column B their complements.

The numbers in A are arranged progressively from the least to the greatest, and hence those in B are also arranged progressively, but in inverse order, from the greatest to the least.

Each number in A and B has, for convenience, been made to consist of 13 digits, and for further convenience, the first 12 digits of every number in A forms a perfect cube.

The roots of these cubes are as follows—

line.	root.	line.	root.	line.	root.	line.	root.
1	4982	9	6815	17	8127	25	8992
2	5035	10	6871	18	8176	26	9128
3	5138	11	7037	19	8261	27	9197
4	5635	12	7168	20	8332	28	9232
5	6023	13	7523	21	8629	29	9463
6	6242	14	7716	22	8677	30	9523
7	6363	15	7821	23	8784	31	9567
8	6447	16	7871	24	8817	32	9615

The numbers have their digits grouped into sets of three for ease of reference, and the last digit of each number is printed apart from the rest, so that it may be readily omitted when necessary to form subcomplements.

22. As the sum of each pair of digits, except the last, is 9, and the sum of the last pair is 10, the number of pairs being 13, the whole sum must be $12 \cdot 9 + 10 = 118$.

23. The sum of each pair of digits thus selected being 9, if there are m pairs, the sum added to m is $9m + m = 10m$.

24. Suppose the number of digits in each number is n, the first number p, the m numbers from A, a_1, $a_2 \ldots a^m$, and consequently their complements from B, $10^n - a_1$, $10^n - a_2 \ldots 10^n - a_m$; and the new number selected by the pupil q. Then the complete sum will be

$$p + a_1 + a_2 + \ldots + a_m + 10^n - a_1 + 10^n - a_2 + \ldots + 10^n - a_m + q$$
$$= m \cdot 10^n + p + q, \text{ as stated in the rule.}$$

25. Adopting the notation in the last article, and taking p as the proof number, the series of numbers on the board will be a_1, $a_2 \ldots a_m$, $10^n - a_1$, $10^n - a_2, \ldots 10^n - a_m + p$, and q being selected by the pupil, the sum of this series and q is $10^n m + p + q$, as before, where $10^n m + p$ is the number represented by P in (**46**). In *Ex.* 1, $p = \frac{1}{9}(10^n - 1)$, in *Ex.* 2, $p = \frac{2}{9}(10^n - 1)$, in *Ex.* 3, $p = \frac{5}{9}(10^n - 1)$. In the other three examples p is negative, and $= -p'$ suppose, where p' is positive and $= \frac{1}{9}(10^n - 1)$ in *Ex.* 4 and 5, and $\frac{2}{9}(10^n - 1)$ in *Ex.* 6. The last number on the board then will be $10^n - a_m + p = 10^n - a_m - p'$ $= (10^n - p') - a_m$, or is found by subtracting a_m from the complement of p'.

26. This is the inverse of *App.* **23.**

27. Here the first result is a—b, and the second
$1\,0^n+(1\,0^n-a)-1\,0^n-b)$ if complements are taken, or
$1\,0^n+(1\,0^n-a-1)-(1\,0^n-b-1)$ if subcomplements are chosen, which in either case $=1\,0^n-(a-b)$ the complement of a—b.

28. For as the corresponding lines in A and B are complementary, if we read them together as parts of one number, and a be the number in A, the new number will be $1\,0^n a+(1\,0^n-a)$, and if we take from this $1\,0^n b+(1\,0^n-b)$, the result is $(a-b)\ (1\,0^n-1)$ which is binonal (*App.* **2.**) If we increase this result by 1, it becomes $(a-b)\ (1\,0^n-1)+1$ or $1\,0^n\ (a-b-1)+\left\{1\,0^n-(a-b-1)\right\}$ and if from this we take $1\,0^n c+(10^n-c)$, or some other complete line, the result is $(1\,0^n-1)\ (a-b-c-1)$, which is also binonal, and so on.

29. This case is simpler than the last, because, as the difference of the binonals is binonal (*App.* **5**), the results are already in a condition for the next subtraction, without requiring the preparation of adding 1.

30. If a_1, $a_2 \ldots a_{m-1}$, a_m be the numbers each containing n digits, then a_m will be the middle number on the board, and the number of complements will be m—1; hence, if M be the n digits of the minuend selected, we must prefix m to M, and the subtraction leads to $m\,.\,10^n+M-a_1-a_2-\ldots-a_{m-1}-a_m-(1\,0^n-a_1)-(1\,0^n-a_2)-\ldots-(1\,0^n-a_{m-1})=1\,0^n+M-a_m$, a positive number, because as a_m only contains n digits, it is less than $1\,0^n$. This contains the rule, because 10^n+M-a_m is the last difference, which, when added to the middle number a_m, produces 10^n+M, or the original minuend m. 10^n+M less $(m-1)\,.\,10^n$, where m—1 is the number of complements.

(**31.**) The lines in A and B being complementary if they be a and $1\,0^n-a$, and M be the multiplier, the sum of the products will be $Ma+M\ (1\,0^n-a)=1\,0^n\,.\,M$, which is the rule.

32. The introduction of the middle zero is only a device to do the two examples of *App.* **31** in one, and the law is the same. The zero prevents any number being carried from the first to the second half of the multiplicand.

33. Let D be the divisor, $D-1=a+b$, and c any number consisting of n digits, so that $(10^n-1)-c$ is its subcomplement. Then the two

dividends are $1\,0^n a+c$ and $1\,0^n b+(1\,0^n-1)-c$; the sum of the results of the two divisions is, therefore,

$$\frac{1\,0^n a+c}{D}+\frac{1\,0^n b+(1\,0^n-1)-c}{D}=\frac{1\,0^n\,(a+b+1)-1}{D}$$
$$=\frac{1\,0^n D-1}{D}=1\,0^n-1+\frac{D-1}{D}$$

That is, the sum of the quotients will be $1\,0^n-1$, or one will be the subcomplement of the other, and the sum of the remainders is the divisor less 1. These two remainders, therefore, are numbers like $a+b$, their sum being $D-1$, while the quotients are subcomplements. Hence a fresh example is formed by prefixing these remainders to the quotient, and using the same divisor. But it may so happen that one of the quotients has one less digit than the other; that is, one may have $n-1$ digits, and the other n, instead of both having n digits. In this case, a 0 must be supplied at the beginning of the smaller, or the 9 omitted at the beginning of the larger quotient, previously to prefixing the remainders. This will make the two new dividends of the forms $1\,0^n a+c$ and $1\,0^n b+(1\,0^n-1)-c$, where c has n digits as before, or $1\,0^{n-1}a+d$ and $1\,0^{n-1}b+(1\,0^{n-1}-1)-d$, where d has $n-1$ digits, and therefore they will be of the right form. But if this precaution be omitted, the new dividends would be of the form $1\,0^{n-1}a+e$, and $1\,0^n b+(10^n-1)-e$, to which the rule will, of course, not apply.

34. The original divisor and dividend being nonundecimal, are of the form $99ab$ and $99ac$, where a is most frequently 1. The remainder is, therefore, of the form $99ad$. Now the process given is that for finding the greatest common measure, $99a$, and, on dividing this original divisor $99ab$, dividend $99ac$, and remainder $99ad$, by $99a$, we obtain b, c, d. Now as $\frac{c}{b}=\frac{99ac}{99ab}$ the quotient in this case will be the same as in the other, but the remainder is now the numerator of the fraction $\frac{d}{b}$ instead of $\frac{99ad}{99ab}$ and is, therefore, $99a$ times less than the former remainder.

35. This rule, which the Author has not yet observed in any English book, is taken from a little pamphlet called "*La Division Abrégée*, ou Méthode Rigoureuse et Facile pour simplifier cette opération de l'Arithmétique; approuvée par l'Académie des Sciences, dans sa séance du 13 Janvier, 1845. Commissaires:—MM. Binet, Cauchy, rapporteur. *Par M. P. Guy*, ancien élève de l'École Polytechnique, capitaine d'artillerie, membre de la légion d'honneur. Ouvrage adopté par l'Université pour l'usage des Colléges, et des Écoles normales

primaires. (Décision du 25 Mars 1845). Paris: L. Mathias, 15, Quai Malaquais, 1845." It is founded on an accurate observation of the partial dividends and remainders in ordinary division. M. Guy shows that if any one unit of the divisor be neglected, an error of 1 in excess may be occasioned in the quotient by the remainder, which may be only 1 less than the divisor; and that by adopting his process (which is very different from the ordinary one) the limits of erroneous increase, due to continually subtracting too little, may be precisely estimated, and prevented from increasing the quotient by 1 if the last divisor is large enough. Hence by his process the error can never exceed 1 in the last digit of the quotient, and will be generally in excess. If the neglected digits were not zeroes, a fresh error, not exceeding 1 in defect, would be introduced, and the two errors being in opposite directions destroy each other, so that the quotient can only have an error of 1, either in excess or defect, in the last digit. The method is also applicable with the same results to the case where digits of both divisor and dividend are cancelled. Thus, if we wished to determine only 3 digits of the quotient of 4 3 6 5 1 5 8 0 0 0 divided by 9 1 2 0 1 0 8, we only require 3 places of the divisor and 4 of the dividend, and the operation will be as in the margin. It is of course impossible to give all the details here.

```
                .  .          4 7 8
912,0108)4865,158000
         7 1 7
           7 9
             7
```

36. Consult *App.* **33**, recollecting that in this case the remainders are neglected.

37. For, taking in the fraction given by the remainder, if a be the divisor and b the first dividend, the quotients will be $\frac{b}{a}$, $\frac{2b}{a}$, $\frac{3b}{a}$, &c., or $\frac{b}{a}$, $\frac{b}{2a}$, $\frac{b}{3a}$, &c. And if we neglect the fraction and prove by division, as shewn in the text, neglecting the remainder of these divisions, the results must be alike.

38. Let a be any number, b its approximate n^{th} root, and c the remainder, then $a = b^n + c$. Hence $z^n a = z^n b^n + z^n c = (zb)^n + z^n c$; that is, the approximate n^{th} root of $z^n a$ is zb, and the remainder $z^n c$. If $n = 2$, this gives the examples for square root, and if $n = 3$ for cube root. In the text z is taken to be $= 2$, but any other number would answer equally well. It may happen that $z^n c$ is so large that $(zb)^n + z^n c = (zb+1)^n + d$, in which case the remainder d is not in a fit form for the application of the rule. We must then reduce the root by 1 and find the corresponding remainder $z^n c$.

39. For, on observing the process for the square root, we perceive that so far as its first figures are concerned it is a pure division; and on

the contracted plan the latter figures are neglected, as not influencing the result. The same is true for cube root.

40. The whole rationale of Horner's process (for which see Prof. De Morgan's Article, INVOLUTION, in the *Penny Cyclopædia*) may be discovered by an attentive study of the following mode of forming the *cube* of $10(10a+b)+c$, or $10^2a+10b+c$, a number of three digits, a, b, c.

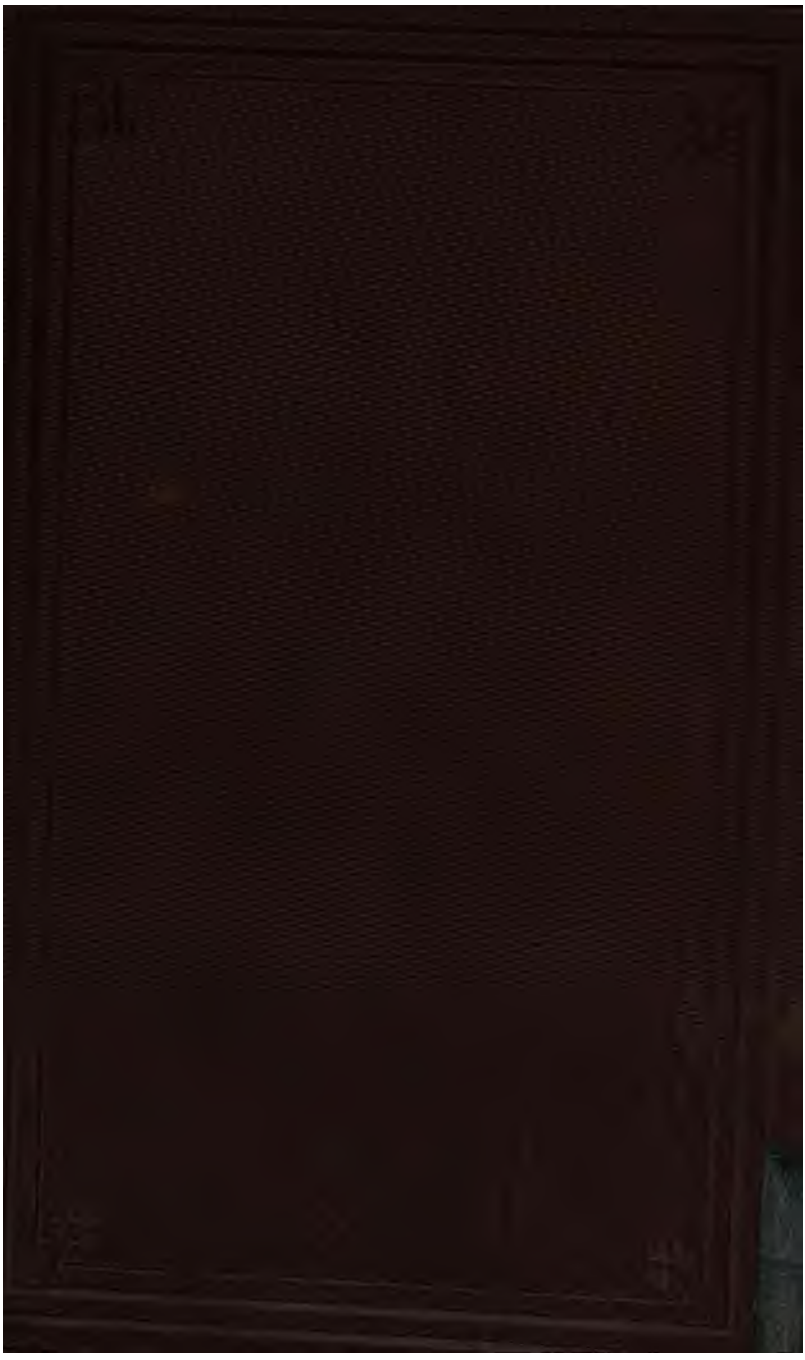
(V) 1 (X)0	(Y)0 0	(Z) 0 0 0
$(C)\ a$	$(E)\ a^2$	$(G)a^3$
$(K)2a$	$(M)3a^2$	$(z)\ a^3\,.\,10^3$
$(N)3a$	$(y)\ 3a^2\,.\,10^2$	
$(x)\ 3a\,.\,10$		
$3a\,.\,10+b$	$3a^2\,10^2+3ab\,.\,10$	$a^3 10^3+3a^2b\,10^2+$
$3a\,.\,10+2b$	$+b^2$	$3ab^2\,10+b^3$
$3a\,.\,10+3b$	$3a^2\,10^2+6ab\,.\,10$	$=(10a+b)^3$
$=3\,(10a+b)$	$+3b^2$	$(10a+b)^3\,.\,10^3$
$3\,(10a+b)\,10$	$=3\,(10a+b)^2$	
	$3\,(10a+b)^2\,10^2$	

The letters V, X, Y, Z, &c., refer to the steps in the example in the text, p. 54. Each expression in each column is formed by multiplying the last expression in the preceding col. by a, and adding it to the expression immediately above it. Thus (M) is (K) multiplied by a and added to (E), or $2a\,.\,a+a^2=3a^2$. (G) is also formed by *addition*, because we are forming a cube by partial steps; but in the text we subtract to form $(G,)$ because we wish to deduct the cube as formed, in order to find the cube next least to the remainder. We complete a stage by multiplying by 10, 10^2, 10^3, respectively.

The first stage leads us to four columns, ending respectively 1, $3a\,.\,10$, $3a^2\,.\,10^2$, $a^3\,.\,10^3$; the second stage leads to four columns, ending 1, $3\,(10a+b)\,.\,10$, $3\,(10a+b)^2\,.\,10^2$, $(10a+b)^3\,.\,10^3$, which only differ from the former by the substitution of $10a+b$ for a. It is evident, therefore, without further calculation, that if we introduce a new addend, c, the completion of the next stage would give us $1, 3\left\{10\,(10a+b)+c\right\}\,.\,10$, $3\left\{10\,(10a+b)+c\right\}^2\,.\,10^2$, $\left\{10\,(10a+b)+c\right\}^3\,.\,10^3$. Hence this is a process by which the cube of 374, for example, might be found by finding successively the cubes of 3, 37, and 374. The application of this process to the extraction of roots, needs no further explanation.

It may, however, be mentioned that any root whatever may be extracted by continuing the process in the requisite number of columns. The reader may practise, therefore, extraction of 5th roots, by taking as two radicands any number and its product by 32. One root will

be half, and the corresponding remainder one thirty-second part of the other. In extracting these higher roots, on account of the large number of digits in the last column but one, the contraction may begin at an early period, and if due attention is paid to the "back" digits, the result may be obtained true to the last digit. In order to do so more easily, the semi-condensed form should be employed. Thus, in the following extraction of the fifth root of 10, where the root is perfectly correct to the last digit inclusive, every figure is put down which the most ordinary computer could require. Only two digits are determined by the full process, the remaining five being found by the contracted method. The advantages of *Horner's Rule* are most clearly displayed by such an example, in which, by an easy process accessible to the humblest computer, a result is obtained that would defy the most skilful calculator who adopted a method similar to that of the ordinary rule for cube root.

			Fifth root of 10	1·5|8 4 8 9 3 *
0	00	000	0000	
1	1	1	1	1
2	3	4	50000	
3	6	10000	81875	900000
4	1000	6375		659375
50	275		131875	
		16375	121250	240625
55	1275	7875		225282
60	300		25312|5	
		24250	2847|8	15343
65	1575	9500		12527
70	325		28160|3	
		33750	2999|5	2816
10075	1900	1848		2520
	350		31159|8	
		35598	15|80	296
	2|250	1896		283
	|06		31317|8	
		37494	15|83	13
	2|31	1944		9
	|06		314|761	
		8|94|38	|316	4
	2|37	|008		
	|06		315|077	
		8|951	|316	
	2|43	|008		
	|06		31|5|893	
		8|959		
	10024|3	|008		
		10|5|9 6 7		

Savill & Edwards, Printers, Chandos Street.